Homework 6-solutions

MATH 361 (due November 4) October 28 2022

Problem 1. Prove that for any $r, s \in \mathbb{R}$, $r \cdot s \in \mathbb{R}$. (You can use problem 6 from HW5).

Solution. The product is defined by cases:

> If $x, y \geq 0$ then $x \cdot y = 0 \cup \{p \cdot q \mid p \in x, q \in y,\ p, q \geq 0\}$.If $x, y < 0$ then $x \cdot y = |x| \cdot |y|$. If $x < 0 \leq y$ or $y < 0 \leq x$ then $x \cdot y = -(|x| \cdot |y|)$

Once we have proven case (1), then since $|x|, |y|$ are non negative, $|x| \cdot |y|$ will be in R and by HW5 problem 6, also $-(|x| \cdot |y|)$ and we will be done. So let us prove case (1). Assume that $x, y \geq 0$ and then $x \cdot y = 0 \cup \{p \cdot q \mid p \in x, q \in y,\ p, q \geq 0\}$. It is clearly non-empty since for example $-1 \in x \cdot y$. Also, if $p^*, q^* > 0$ bound x, y respectively, then for every $z \in x \cdot y$, either $z < 0 < p^* q^*$, or $z = p \cdot q$ for some $p, q \geq 0$ and $p \in x$ and $q \in y$. It follows that $p < p^*$ and $q < q^*$ which in turn implies (since we are dealing with positive rationals) that $p \cdot q < p^* \cdot q^*$. Hence $p^* \cdot q^*$ bounds $x \cdot y$. To see it is downward closed, let $t < z \in x\dot{y}$. if $t \leq 0$ then it is clearly in $x \cdot y$. Otherwise, $0 < t \leq z$ and therefore $z = p \cdot q$ for some $p, q > 0$ rationals. where $p \in x$ and $q \in y$. Let $q' = \frac{z}{p}$. Then $z = p \cdot q'$ and since $p \cdot q' \leq p \cdot q$, we have that $q' \leq q$ (since q, q', p are all positive). Since y is a Dedekind cut, $q' \in y$ and therefore $z = p \cdot q' \in x \cdot y$. Finally we need to prove that $x \cdot y$ has no last element. Let $q \in x \cdot y$. If $q < 0$ then $q\ q < \frac{q}{2} < 0$ hence $\frac{q}{2} \in x \cdot y$. If $q \geq 0$, then there are $0 \leq p_1, p_2$, $p_1 \in x$ and $p_2 \in y$ such that $q = p_1 \cdot p_2$. Since x, y are dedekinf cuts, there are $p_1 < p_1' \in x$ and $p_2 < p_2' \in y$. Since they are all positive, $p_1 \cdot p_2 < p_1' \cdot p_2' \in x \cdot y$ as wanted.

Problem 2. For each of the following statements provide an appropriate function (no need to prove that your functions satisfy the required properties):

1. $\mathbb{R} \approx \mathbb{R} \setminus \{0\}$.

   Solution. $f : \mathbb{R} \to \mathbb{R} \setminus \{0\}$

$$f(x) = \begin{cases} x - 1 & x \in \mathbb{N}^+ \\ x & o.w. \end{cases}$$

2. $\mathbb{Z} \approx \mathbb{N}_{even} \setminus \{0, ..., 2023\}$.

   Solution. Define $f : \mathbb{Z} \to \mathbb{N}_{even} \setminus \{0, ..., 2023\}$ by

$$f(n) = \begin{cases} 2024 + 4|n| & n \leq 0 \\ 2022 + 4n & n > 0 \end{cases}$$

3. $\mathbb{N} \times \mathbb{N} \leq {}^{\mathbb{N}}\{0, 1\}$

   Solution. We saw in class one example $f(\langle n, m\rangle) = 00000...\underset{n^{\text{th}}\text{ place}}{1}0...0\underset{n+m^{\text{th}}\text{ place}}{1}0....$ Formally, $f(\langle n, m\rangle) : \mathbb{N} \to \{0, 1\}$ is defined

$$f(\langle n, m\rangle)(k) = \begin{cases} 1 & k \in \{n, n + m\} \\ 0 & o.w. \end{cases}$$

4. $\left\{f \in {}^{\mathbb{R}}\mathbb{R} \mid \exists i \in \{0, 1\},\ \forall x \in \mathbb{R} \setminus \mathbb{Q},\ f(x) = i\right\} \approx \{0, 1\} \times {}^{\mathbb{Q}}\mathbb{R}$.

   Solution. Denote the set by A. $F : A \to \{0, 1\} \times {}^{\mathbb{Q}}\mathbb{R}$ defined by

$$F(f) = \langle f(\sqrt{2}), f \restriction \mathbb{Q}\rangle$$

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Problem 3. Prove that

$$\Big\{X \in P(\mathbb{N}) \mid \mathbb{N}_{even} \subseteq X\Big\} \approx P(\mathbb{N})$$

[Hint: First find a function from $P(\mathbb{N})$ to $P(\mathbb{N}_{odd})$]

Solution. Like in the proof that $A \approx B \Rightarrow P(A) \approx P(B)$ we fix a bijection $f : \mathbb{N} \to \mathbb{N}_{odd}$, for example $f(n) = 2n + 1$, then $F(X) = f''X$ is a bijection from $P(\mathbb{N})$ to $P(\mathbb{N}_{odd})$. Explicitly, $F(X) = \{2n + 1 \mid n \in X\}$. Now define $G : P(\mathbb{N}) \to \{X \in P(\mathbb{N}) \mid \mathbb{N}_{even} \subseteq X\}$ defined by $G(X) = \mathbb{N}_{even} \cup \{2n + 1 \mid n \in \mathbb{N}\}$. Check that this is a bijection.

Problem 4. Let $C(\mathbb{R})$ be the set of all continuous function $f : \mathbb{R} \to \mathbb{R}$. Prove that

$$C(\mathbb{R}) \leq {}^{\mathbb{Q}}\mathbb{R}$$

[Hint: use that fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ to prove that the restriction function $G : C(\mathbb{R}) \to {}^{\mathbb{Q}}\mathbb{R}$ defined by $G(f) = f \restriction \mathbb{Q}$ is one-to-one.]

Solution. Let $G : C(\mathbb{R}) \to {}^{\mathbb{Q}}\mathbb{R}$ defined by $G(f) = f \restriction \mathbb{Q}$, let us prove that it is one-to-one. Suppose that f, g are two continuous functions, such that $f \restriction \mathbb{Q} = g \restriction \mathbb{Q}$. We need to prove $f = g$. Let $x \in \mathbb{R}$, by density of the rationals we can find a sequence $(q_n)_{n=0}^{\infty}$ of rationals, such that $\lim_{n\to\infty} q_n = x$, then for each n, $f(q_n) = g(q_n)$ (since $f \restriction \mathbb{Q} = g \restriction \mathbb{Q}$). By continuity,

$$f(x) = \lim_{n\to\infty} f(q_n) = \lim_{n\to\infty} g(q_n) = g(x)$$

Problem 5. Prove that if $A \approx B$ and $C \approx D$ then $A \times C \approx B \times D$.

Solution. Let $f : A \to B$ and $g : C \to D$ be bijections. Define $h : A \times C \to B \times D$ $h(\langle a, c) = \langle f(a), g(c)\rangle$. Prove that h is one-to-one. Let us prove for example that h is onto. Let $\langle b, d\rangle \in B \times D$. Since f, g are onto, there are $a \in A$ and $c \in C$ such that $f(a) = b$ and $g(c) = d$. Then $\langle a, c\rangle \in A \times C$ and $h(\langle a, c\rangle) = \langle f(a), g(c)\rangle = \langle b, d\rangle$.

Problem 6. Prove that for every $\alpha < \beta$ real numbers $(\alpha, \beta) \approx (0, 1)$. [Hint: First stretch/shrink $(0, 1)$ to have length $\beta - \alpha$, then shift it by $+c$ as we did in class.]

Solution. Define $f : (0, 1) \longrightarrow (\alpha, \beta)$ by $f(x) = (\beta - \alpha)x + \alpha$. It is not hard to check that f is one-to-one and onto.

Additional problems

Problem 7. Show that $x \cdot (y + z) = x \cdot y + x \cdot x$ for every $x, y, z \in \mathbb{R}$.

Problem 8. Show that for every $n > 0$, $\mathbb{N}^n \approx \mathbb{N}$. [Hint: Induction. you can $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$.]

Problem 9. Show that ${}^{\mathbb{N}}\{0, 1\} \times {}^{\mathbb{N}}\{0, 1\} \approx {}^{\mathbb{N}}\{0, 1\}$. [Hint: see HW2 problem 5.]