Homework 5

MATH 361 (due October 27) October 20, 2023

Problem 1. 1. Prove that addition in $\mathbb{Z}$ is commutative. [Hint: use the fact that addition in $\mathbb{N}$ is already known to be commutative]

2. Recall that a natural number n is identified with $n = [\langle n, 0]_{\sim_Z}$ and $-n := [\langle 0, n\rangle]_{\sim_Z}$. Prove that $n + (-n) = 0$.

Problem 2. Prove that for every $[\langle n, m\rangle]_{\sim_Q} \in \mathbb{Q}$ there is $n', m' \in \mathbb{Z}$ such that $m' > 0$ and $[\langle n, m\rangle]_{\sim_Q} = [\langle n', m'\rangle]_{\sim_Q}$.

Problem 3. Prove that $(0, 1) \cap \mathbb{Q}$ with the regular order is ismorphic to $\mathbb{Q}$.[Hint: Apply Cantor's theorem, no need to prove htat $(0, 1) \cap \mathbb{Q}$ is countable.]

Problem 4. Prove that the union of Dedekind cuts is a Dedekind cut.

Problem 5. Prove that the function $f(q) = \{q' \in \mathbb{Q} \mid q' < q\}$ is an embedding of $\mathbb{Q}$ in $\mathbb{R}$.

Problem 6. Recall that for $x \in \mathbb{R}$ we define:

$$-x = \{q \in \mathbb{Q} \mid \exists s > q,\ -s \notin x\}.$$

Prove that $-x \in \mathbb{R}$.

1 Additional problems

Problem 7. In this problem we are going to prove Cantor's theorem for dense linear orders with no least and last element. Recall that the theorem is:

Suppose that $\langle A, <_A\rangle$ is a linearly ordered set such that:

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(a) A is countable.

(b) the order $<_A$ is dense in itself i.e. for every $a_1, a_2 \in A$ if $a_1 <_A a_2$ then there is $a_3 \in A$ such that $a_1 <_A a_3 <_A a_2$.

(c) There is no least element in A, namely for every $a \in A$ there is $b \in A$ such that $b <_A a$.

(d) There is no last element, namely, for every $a \in A$ there is $b \in A$ such that $b >_A a$.

$$\text{Then } \langle A, <_A \rangle \simeq \langle \mathbb{Q}, < \rangle.$$

To prove the theorem let us construct an isomorphism $f : \mathbb{Q} \to A$.

- Step 1: Enumerate $A = \{a_n \mid n \in \mathbb{N}\}$ and $\mathbb{Q} = \{q_n \mid n \in \mathbb{N}\}$, **explain why is this possible**.
- step 2: Define recursively a sequence of pairs $\langle x_n, y_n \rangle$ in $\mathbb{Q} \times A$.

  (I) Let $\langle x_0, y_0 \rangle = \langle x_1, y_1 \rangle = \langle q_0, a_0 \rangle$.

  (II) (For clarity reasons, let us do $n = 2, 3$).

  (i) If $q_1 > q_0$ pick $a_m >_A a_0$.

  (ii) If $q_1 < q_0$ pick $a_m <_A a_0$.

  Explain why there must be such an a_m. Define $\langle x_2, y_2 \rangle = \langle q_1, a_1 \rangle$. If $a_m = a_1$, define also $\langle x_3, y_3 \rangle = \langle q_1, a_1 \rangle$. Otherwise, consider a_1,

  (i) If $a_1 <_A \min(a_0, a_m)$ pick $q_k < \min(q_0, q_1)$.

  (ii) If $a_1 <_A \max(a_0, a_m)$ pick $q_k > \max(q_0, q_1)$.

(iii) If $\min(a_0, a_m) <_A a_1 <_A \max(a_0, a_m)$ pick $\min(q_0, q_1) < q_k < \max(q_0, q_1)$.

Explain why these are the only three possibilities and why there must be such an q_k. Define $\langle x_3, y_3 \rangle = q_k, a_1\rangle$.

(III) Suppose that $\langle x_0, y_0\rangle, ..., \langle x_{2n-1}, y_{2n-1}\rangle$ have been defined so that $x_i < x_j$ if and only if $y_i <_A y_j$ and consider q_n:

(i) If $q_n = x_i$ for some $i < 2n$, define $\langle x_{2n}, y_{2n}\rangle = \langle x_i, y_i\rangle$.

(ii) If $q_n < \min\{x_1, ..., x_{2n-1}\}$ pick $a <_A \min\{y_1, ..., y_{2n-1}\}$.

(iii) If $q_n > \max\{x_1, ..., x_{2n-1}\}$ pick $a >_A \max\{y_1, ..., y_{2n-1}\}$.

(iv) Otherwise let x_i be the maximal among $x_1, ..., x_{2n-1}$ which is below q_n and let x_j be minimal among $x_1, ..., x_{2n-1}$ which is above q_n (**why are there such x_i and x_j**). Then $x_i < q_n < x_j$ and pick $y_i <_A a <_A y_j$ (**why can we find such a?**)

Define $\langle x_{2n}, y_{2n}\rangle = \langle q_n, a\rangle$

Fill up the definition of $\langle x_{2n+1}, y_{2n+1}\rangle$ considering now a_n. This should be very similar to the above

- Step 3: Define $f = \{\langle x_n, y_n\rangle \mid n \in \mathbb{N}\}$ **Prove that $f : \mathbb{Q} \to A$ is a function. The totality should follow from the fact that $\mathbb{Q} = \{q_n \mid n \in \mathbb{N}\}$.**

- Step 4: **Prove that f is order preserving. Use the recursive definition.**

- Step 5: **Prove that F is a bijection (1-1 should follow in general for order preserving functions and onto follows from the fact that**

$A = \{a_n \mid n \in \mathbb{N}\}$**)**