Homework 5-Sols

MATH 361 (due October 27) October 20, 2023

Problem 1. 1. Prove that addition in $\mathbb{Z}$ is commutative. [Hint: use the fact that addition in $\mathbb{N}$ is already known to be commutative]

Solution. By definition of addition,

$$[\langle n, m\rangle]_{\sim_Z} + [\langle n', m'\rangle]_{\sim_Z} = [\langle n + n', m + m'\rangle]_{\sim_Z}.$$

Since Addition in $\mathbb{N}$ is commutative,

$$[\langle n' + n, m + m'\rangle]_{\sim_Z} = [\langle n', m'\rangle]_{\sim_Z} + [\langle n, m\rangle]_{\sim_Z}.$$

2. Recall that a natural number n is identified with $n = [\langle n, 0\rangle]_{\sim_Z}$ and $-n := [\langle 0, n\rangle]_{\sim_Z}$. Prove that $n + (-n) = 0$.

Solution. $n + (-n) = [\langle n, 0\rangle]_{\sim_Z} + [\langle 0, n\rangle]_{\sim_Z} = [\langle n, n\rangle]_{\sim_Z} =^* [\langle 0, 0\rangle]_{\sim_Z}$ to see (*), note that $n + 0 = 0 + n$ and therefore $\langle 0, 0\rangle \sim_Z \langle n, n\rangle$ which implies that $[\langle 0, 0\rangle]_{\sim_Z} = [\langle n, n\rangle]_{\sim_Z}$

Problem 2. Prove that for every $[\langle n, m\rangle]_{\sim_Q} \in \mathbb{Q}$ there is $n', m' \in \mathbb{Z}$ such that $m' > 0$ and $[\langle n, m\rangle]_{\sim_Q} = [\langle n', m'\rangle]_{\sim_Q}$.

Solution. If $m > 0$ just take $n' = n$ and $m' = m$. If $m < 0$, take $n' = -n$ and $m' = -m$. Then $m' > 0$ and $nm' = n(-m) = (-n)m = n'm$. Hence $[\langle n, m\rangle]_{\sim_Q} = [\langle n', m'\rangle]_{\sim_Q}$.

Problem 3. Prove that $(0, 1) \cap \mathbb{Q}$ with the regular order is ismorphic to $\mathbb{Q}$.[Hint: Apply Cantor's theorem, no need to prove htat $(0, 1) \cap \mathbb{Q}$ is countable.]

Solution. Done in class.

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Problem 4. Prove that the union of Dedekind cuts is a Dedekind cut.

Solution. There is a typo here, this should have been that every non-empty(!) bounded(!) union of Dedekind cuts is a Dedekind cut. Let $\mathcal{F} \subseteq \mathbb{R}$ be a set of Dedekind cuts such that $r \in \mathbb{R}$ bounded $\mathcal{F}$. Let us prove that $\cup\mathcal{F}$ is a Dedekind cut. Since $\mathcal{F}$ is non-empty, there is $s \in \mathcal{F}$, and $s \neq \emptyset$ since s is a D.cut. Since $s \cup \mathcal{F}$, $\cup\mathcal{F} \neq \emptyset$. To see that $\cup\mathcal{F}$ is downward closed, let $x \in \cup\mathcal{F}$ and $y < x$. Then there is $s \in \mathcal{F}$ such that $x \in s$. Since s is a D.cut, $y \in s$ and therefore $y \in \cup\mathcal{F}$. To see that $\cup\mathcal{F}$ has no last element, let $x \in \cup\mathcal{F}$, then there is $s \in \mathcal{F}$ such that $x \in s$. Since s is a D.cut, it had no last element and therefore there is $x < y \in s$. But then $y \in \cup\mathcal{F}$ and therefore $\mathcal{F}$ has no last element. Finally, to see that $\cup\mathcal{F}$ is bounded, since $\mathcal{F}$ is bounded, there is $r \in \mathbb{R}$ such that for every $s \in \mathcal{F}$, $s \subseteq r$. Since r is bounded in $\mathbb{Q}$, there is $q \in \mathbb{Q}$ such that for every $p \in r$, $p < q$. Hence for every $x \in \cup\mathcal{F}$, there is $s \in \mathcal{F}$ such that $x \in s$ and therefore $x \in r$ and hence $x < q$. It follows that $\cup\mathcal{F}$ is bounded in $\mathbb{Q}$.

Problem 5. Prove that the function $f(q) = \{q' \in \mathbb{Q} \mid q' < q\}$ is an embedding of $\mathbb{Q}$ in $\mathbb{R}$.

Solution. To see it is one-to-one, let $q_1 < q_2$, then by density of the rationals there is $q_1 < q < q_2$, then $q \in f(q_2)$ but $q \notin f(q_1)$. Hence $f(q_1) \neq f(q_2)$. Prove that it is order preserving.

Problem 6. Recall that for $x \in \mathbb{R}$ we define:

$$-x = \{q \in \mathbb{Q} \mid \exists s > q,\ -s \notin x\}.$$

Prove that $-x \in \mathbb{R}$.

Solution. To see that $-x$ is not empty, take any $t \notin x$ (which exists since x is bounded in $\mathbb{Q}$) and consider $q = -(t+1)$. Then for $s = -t$, we have that $s > q$ and $-s = t+1 \notin x$ (since otherwise also $t \in x$ by downward closure of D.cuts). Hence $q \in -x$. To see that it is non-empty. It is easy to see it is dowmward closed. To see it as no last element, let $q \in -x$, then there is $s > q$ such that $-s \notin x$. By density of the rationals, find $q < q' < s$, then $q' \in x$ as well as witnessed by the same s. It remains to see that $-x$ is bounded. Take any $p \in x$, then $-p$ is not in x since otherwise there is $s > p$ such that $-s \notin x$, but $-s < -p$ and $-p \in x$ so $-s \in x$ by downwards closure. So $-p > q$ for every $q \in -x$ (since otherwise $p \leq q$ and we already showed that $-x$ is downwards closed so $p \in -x$, contradiction).

1 Additional problems

Problem 7. In this problem we are going to prove Cantor's theorem for dense linear orders with no least and last element. Recall that the theorem is:

Suppose that $\langle A, <_A \rangle$ is a linearly ordered set such that:

(a) A is countable.

(b) the order $<_A$ is dense in itself i.e. for every $a_1, a_2 \in A$ if $a_1 <_A a_2$ then there is $a_3 \in A$ such that $a_1 <_A a_3 <_A a_2$.

(c) There is no least element in A, namely for every $a \in A$ there is $b \in A$ such that $b <_A a$.

(d) There is no last element, namely, for every $a \in A$ there is $b \in A$ such that $b >_A a$.

$$\text{Then } \langle A, <_A\rangle \simeq \langle \mathbb{Q}, <\rangle.$$

To prove the theorem let us construct an isomorphism $f : \mathbb{Q} \to A$.

- Step 1: Enumerate $A = \{a_n \mid n \in \mathbb{N}\}$ and $\mathbb{Q} = \{q_n \mid n \in \mathbb{N}\}$, **explain why is this possible**.

- step 2: Define recursively a sequence of pairs $\langle x_n, y_n\rangle$ in $\mathbb{Q} \times A$.

  (I) Let $\langle x_0, y_0\rangle = \langle x_1, y_1\rangle = \langle q_0, a_0\rangle$.

  (II) (For clarity reasons, let us do $n = 2, 3$).

    (i) If $q_1 > q_0$ pick $a_m >_A a_0$.

    (ii) If $q_1 < q_0$ pick $a_m <_A a_0$.

  Explain why there must be such an a_m. Define $\langle x_2, y_2\rangle = \langle q_1, a_1\rangle$. If $a_m = a_1$, define also $\langle x_3, y_3\rangle = \langle q_1, a_1\rangle$. Otherwise, consider a_1,

    (i) If $a_1 <_A \min(a_0, a_m)$ pick $q_k < \min(q_0, q_1)$.

    (ii) If $a_1 <_A \max(a_0, a_m)$ pick $q_k > \max(q_0, q_1)$.

    (iii) If $\min(a_0, a_m) <_A a_1 <_A \max(a_0, a_m)$ pick $\min(q_0, q_1) < q_k < \max(q_0, q_1)$.

  Explain why these are the only three possibilities and why there must be such an q_k. Define $\langle x_3, y_3\rangle = q_k, a_1\rangle$.

  (III) Suppose that $\langle x_0, y_0\rangle, \ldots., \langle x_{2n-1}, y_{2n-1}\rangle$ have been defined so that $x_i < x_j$ if and only if $y_i <_A y_j$ and consider q_n:

(i) If $q_n = x_i$ for some $i < 2n$, define $\langle x_{2n}, y_{2n} \rangle = \langle x_i, y_i \rangle$.

(ii) If $q_n < \min\{x_1, ..., x_{2n-1}\}$ pick $a <_A \min\{y_1, ..., y_{2n-1}\}$.

(iii) If $q_n > \max\{x_1, ..., x_{2n-1}\}$ pick $a >_A \max\{y_1, ..., y_{2n-1}\}$.

(iv) Otherwise let x_i be the maximal among $x_1, ..., x_{2n-1}$ which is below q_n and let x_j be minimal among $x_1, ..., x_{2n-1}$ which is above q_n (**why are there such** x_i **and** x_j). Then $x_i < q_n < x_j$ and pick $y_i <_A a <_A y_j$ (**why can we find such** a**?**)

Define $\langle x_{2n}, y_{2n} \rangle = \langle q_n, a \rangle$

Fill up the definition of $\langle x_{2n+1}, y_{2n+1} \rangle$ **considering now** a_n**. This should be very similar to the above**

- Step 3: Define $f = \{\langle x_n, y_n \rangle \mid n \in \mathbb{N}\}$ **Prove that** $f : \mathbb{Q} \to A$ **is a function. The totality should follow from the fact that** $\mathbb{Q} = \{q_n \mid n \in \mathbb{N}\}$**.**

- Step 4: **Prove that** f **is order preserving. Use the recursive definition.**

- Step 5: **Prove that** F **is a bijection (1-1 should follow in general for order preserving functions and onto follows from the fact that** $A = \{a_n \mid n \in \mathbb{N}\}$**)**