Homework 4-Sols

MATH 361 (due October 20) October 13, 2023

Problem 1. Prove that rational addition defined by:

$$[\langle n, m\rangle]_{\sim_Q} + [\langle n', m'\rangle]_{\sim_Q} = [\langle nm' + n'm, mm'\rangle]_{\sim_Q}$$

does not depend on the choice of representatives.

Solution. Suppose $[\langle n, m\rangle]_{\sim_Q} = [\langle n_1, m_1\rangle]_{\sim_Q}$ and $[\langle n', m'\rangle]_{\sim_Q} = [\langle n'_1, m'_1\rangle]_{\sim_Q}$ we need to prove that $[\langle nm' + n'm, mm'\rangle]_{\sim_Q} = [\langle n_1m'_1 + n'_1m_1, m_1m'_1\rangle]_{\sim_Q}$. By assumption,

$$(I)\ \ nm_1 = n_1m \text{ and } (II)\ \ n'm'_1 = n'_1m'$$

We need to prove that $(nm' + n'm)m_1m'_1 = (n_1m'_1 + n'_1m_1)mm'$. Opening the brackets, this reduces to

$$(*)\ \ nm'm_1m'_1 + n'mm_1m'_1 = n_1m'_1mm' + n'_1m_1mm'$$

Multiplying (I) by $m'm'_1$ and (II) by mm_1 wee have

$$nm'm_1m'_1 = n_1m'_1mm' \ \text{ and } \ n'mm_1m'_1 = n'_1m_1mm'$$

add those qualities to deduce that $(*)$ holds.

Problem 2. For two function $f, g \in {}^{\mathbb{N}}\mathbb{N}$ deinfe

$$f \le^* g \iff \exists N \forall n \ge N,\ f(n) \le g(n)$$

1. Prove that $\le^*$ is not anti-symmetric.

   Solution. For example $f_1(n) = 0$ and $f_2(n) = \begin{cases} 1 & n = 0 \\ 0 & n > 0 \end{cases}$ are tw distinct functions (Since $f_1(0) = 0 \neq 1 = f_2(0)$) and for every $n \ge 1$ $f_1(n) = 0 = f_2(n)$. So $f_1 \le^* f_2$ and $f_2 \le^* f_1$ but $f_1 \neq f_2$.

2. Let

$$E = \{\langle f, g\rangle \in ({}^{\mathbb{N}}\mathbb{N})^2 \mid \exists N \forall n \geq N, f(n) = g(n)\}$$

Prove that E is an equivalence relation.

Solution. Proved in class

3. Prove that the relation $[f]_E \leq^* [g]_E$ iff $f \leq^* g$ does not depend on the choice of representatives and partially orders ${}^{\mathbb{N}}\mathbb{N}/E$.

Solution. Suppose that $[f']_E = [f]_E$ and $[g']_E = [f]_E$. We need to prove that $f \leq^* g$ if and only if $f' \leq^* g'$. By symmetry, it suffices to prove $f \leq^* g \Rightarrow f' \leq^* g'$. Suppose there is N such that $\forall n \geq N$, $f(n) \leq g(n)$. Since $[f]_E = [f']_E$ and $[g]_E = [g']_E$ there are N_1, N_2 such that for every $n \geq N_1$ $f(n) = f'(n)$ and for every $n \geq N_2$, $g(n) = g'(n)$. Let $N^* = \max(N, N_1, N_2)$. Then for every $n \geq N^*$,

$$f'(n) = f(n) \leq g(n) = g'(n),$$

hence $f' \leq^* g'$. To see that $\leq^*$ partially orders ${}^{\mathbb{N}}\mathbb{N}$, it is clearly reflexive and transitive. To see it is strongly anti-symmetric, suppose that $[f]_E \leq^* [g]_E$ and $[g]^*_E[f]_E$, we need to prove that $[f]_E = [g]_E$. Translating this, we have that $f \leq^* g$ and $g \leq^* f$ and we need to prove that fEg. There are N_1, N_2 such that for every $n \geq N_1$, $f(n) \leq g(n)$ and for every $n \geq N_2$, $g(n) \leq f(n)$. Hence for every $n \geq \max(N_1, N_2)$, $f(n) \leq g(n) \wedge g(n) \leq f(n) \Rightarrow f(n) = g(n)$. It follows that there if N such that for every $n \geq N$ $f(n) = g(n)$, namely fEg, as desired.

Problem 3. Prove or disprove $\langle \mathbb{N}, < \rangle \simeq \langle \mathbb{N} \times \mathbb{N}, <_{Lex} \rangle$

Solution. Disprove! Suppose toward a contradiction that $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is an isomorphism (order preserving bijection). Since f is supposed to be onto, there is n such that $f(n) = \langle 1, 0\rangle$. Since f is order preserving, for every $m < n$, $f(m) <_{LEX} f(n) = \langle 1, 0\rangle$ and therefore there is k such that $f(n-1) = \langle 0, k\rangle$. Again since f is onto, there is $t \in \mathbb{N}$ such that $f(t) = \langle 0, k+1\rangle$, however, $\langle 0, k\rangle <_{LEX} \langle 0, k+1\rangle <_{LEX} \langle 1, 0\rangle$ and so $f(n-1) < f(t) < f(n)$. Since f is order preserving, $n - 1 < t < n$, contradiction to the fact that there are no natural numbers between $n - 1$ and n. Note that this prof does not work if $n = 0$. Prove the case $n = 0$ yourself!

Problem 4. Prove that for all $m \in \mathbb{N}$, either $m = \emptyset$ or $\emptyset \in m$. [Hint: Show that $S = \{m \in \mathbb{N} \mid m = \emptyset \text{ or } \emptyset \in m\}$ is inductive.]

Solution. Let us prove that S is an inductive set. Indeed, $0 = \emptyset \in S$. Suppose that $n \in S$, if $n = 0$, then $\emptyset = 0 \in 0 \cup \{0\} = 1$ and therefore $1 \in S$. Otherwise, by definition of S, $\emptyset \in m$. It follows that $\emptyset \in m \cup \{m\}$ (by definition of union) and therefore S is inductive. By the induction theorem, $S = \mathbb{N}$.

Problem 5. Given distributively in the natural numbers, prove that the multiplication is associative

Solution. Let us prove by induction on k that $(m \cdot n) \cdot k = m \cdot (n \cdot k)$. For $k = 0$ we have $(m \cdot n) \cdot 0 = 0$ by definition of multiplication. and also $n \cdot 0 = 0$ for the same reason. Hence $m \cdot (n \cdot 0) = m \cdot 0 = 0$. Suppose that this holds for k, and let us prove for $k + 1$.

$$(n\cdot m)\cdot(k+1) =^{*} (n\dot{m})\cdot k+(n\cdot m) =^{**} n\cdot(m\cdot k)+n\cdot m =^{***} n\cdot(m\cdot k+m) =^{*} n\cdot(m\cdot(k+1))$$

(*)- by the recursive definition $x \cdot (k+1) = x \cdot k + x$.

(**)- The induction hypothesis

(***)- since we assume distributively.

Problem 6. Prove that $(n \cdot m)^k = n^k \cdot m^k$.

Solution. By induction on k, for $k = 0$ we have

$$(n \cdot m)^0 = 1$$

by the definition of exponent and also $n^0 = 1 = m^0$. Now $n^0 \cdot m^0 = 1 \cdot 1 = 1 \cdot (0+1) = 1 \cdot 0 + 1 = 0 + 1 = 1$. Assume this holds for k and let us prove for $k+1$.

$$(n \cdot m)^{k+1} =^* (n \cdot m)^k \cdot (n \cdot m) =^{**} (n^k \cdot m^k) \cdot (n \cdot m) =^{***} (n^k \cdot n) \cdot (m^k \cdot m) =^* n^{k+1} \cdot m^{k+1}$$

(*)- recursive definition $x^{k+1} = x^k \cdot x$.

(**)- induction hypothesis.

(***)- associativity and commutativity of multiplication