Homework 2

Problem 1. Prove or disprove the following items:

1. If $f : A \to B$ is injective, then for every $X \subseteq A$, $f \restriction X$ is injective.
2. If $f : A \to B$ is surjective, then for every $X \subseteq A$, $f \restriction X$ is surjective.

Solution.
1. *The statement is true. Proof: Let $f : A \to B$ be an injective function, and $X \subseteq A$. We want to prove that $f \restriction X$ is injective. So, let $x_1, x_2 \in X$ such that $(f \restriction X)(x_1) = (f \restriction X)(x_2)$. As $\forall x \in X, (f \restriction X)(x) = f(x)$, this is equivalent to proving that $f(x_1) = f(x_2)$. By our assumption, f is injective, so $x_1 = x_2$. Therefore, $(f \restriction X)$ is injective.*
2. *The statement is false. For example consider the identity function $id_{\{1,2\}}$ and $X = \{1\}$. Then $f \restriction \{1\}$ is not onto $\{1, 2\}$.*

Problem 2. Prove that if $f : A \to B$ is a function such that for some $X \subsetneq A$, $f \restriction X : X \to B$ is onto B, then f is not injective.

Solution. *Since $X \subsetneq A$, there is $a \in A \setminus X$. Let $b = f(a) \in B$. Since $f \restriction X$ is surjective, there is $x \in X$ such that $f(x) = b$. Note that $a \neq x$ as $x \in X$ and $a \notin X$, and also $f(a) = b = f(x)$. It follows that f is not injective.*

Problem 3. For each of the following functions, determine if it is injective/ surjective and prove your answer for two of the items which are not the first once.

1. $f_1 : \mathbb{R} \to \mathbb{R}$, defined by $f_1(x) = 5x - x^2$.
2. $f_2 : \mathbb{R} \to P(\mathbb{R})$, defined by $f_2(x) = \{x^2\}$.
3. $f_3 : P(\mathbb{R}) \to P(\mathbb{N})$, defined by $f_3(x) = x \cap \mathbb{N}$.

4. $f_4 : P(\mathbb{N}) \to \mathbb{N}$, defined by $f_4(x) = \begin{cases} \min(x) & 4 \in x \\ 0 & else \end{cases}$.

5. $f_5 : P(\mathbb{R}) \to P(\mathbb{N}) \times P(\mathbb{Z}) \times P(\mathbb{Q})$, defined by

$$f_5(X) = \langle X \cap \mathbb{N}, X \cap \mathbb{Z}, X \cap \mathbb{Q} \rangle$$

6. $f_6 : P(\mathbb{N}) \to P(\mathbb{N}_{even}) \times P(\mathbb{N}_{odd})$ defined by $f_6(X) = \langle \{2n \mid n \in X\}, \{2n+1 \mid n \in X\} \rangle$.

Solution. 1. *f_1 is nor injective nor surjective. Proof:*

   (a) *$f_1(0) = 0 = f_1(5)$. Clearly $0 \neq 5$, so f_1 is not injective.*

   (b) *There exists $y \in \mathbb{R}$ such that $\forall x \in \mathbb{R}, f(x) \neq y$. In particular, let $y = 8$. The equation $8 = 5x - x^2$ has no real solution. So, $\forall x \in \mathbb{R}, f(x) \neq 8$. Therefore, f_1 is not surjective.*

2. *f_2 is nor injective nor surjective. Proof:*

   (a) *$f_2(1) = \{1\} = f_2(-1)$. Clearly $1 \neq -1$, so f_2 is not injective.*

   (b) *Consider the set $\{1, 2\} \subseteq P(\mathbb{R})$. Note that for all x, $f_2(x)$ has only one element, but $\{1, 2\}$ has two elements. So $\forall x \in \mathbb{R}, fx^2 \neq \{1, 2\}$, and thus f_2 is not surjective.*

3. *f_3 is surjective but not injective. Proof:*

   (a) *$f_3(\{1.5\}) = \emptyset = f_3(\{1.1\})$, but $\{1.5\} \neq \{1.1\}$. Therefore, f_3 is not injective.*

   (b) *Let $Y \in P(\mathbb{N})$, and $X = Y$. Then $X \subseteq P(\mathbb{R})$, and $f_3(X) = X \cap \mathbb{N} = X$. Therefore, f_3 is surjective.*

4. *f_4 is not injective or surjective. Proof:*

   (a) $f_4(\{1\}) = 0 = f_4(\{2\})$, *but* $\{1\} \neq \{2\}$. *Therefore,* f_4 *is not injective.*

   (b) *Let* y *be a natural number greater than 4, and let* $X \subseteq \mathbb{N}$. *Cases:*

      i. $4 \in X$. *Then* $\min(X) \leq 4$, *and so* $f_4(X) < y$.

      ii. $4 \notin X$. *Then* $f_4(X) = 0 \neq y$.

      Therefore, f_4 *is not surjective.*

5. *f_5 is nor injective nor surjective. Proof:*

   (a) $f_5(\{\pi\}) =< \emptyset, \emptyset, \emptyset >= f_5(\{\sqrt{2}\})$, *but* $\{\pi\} \neq \{\sqrt{2}\}$. *Therefore,* f_5 *is not injective.*

   (b) *Let* $Y =< \{1\}, \{-1\}, \{\frac{1}{2}\} >$. *Towards a contradiction, suppose* f_5 *is surjective. Then there exists some* $X \in P(\mathbb{R})$ *such that* $f_5(X) = Y$. *By the definition of* f, *for some* $N \subseteq \mathbb{N}, X \cap \mathbb{N} = \{1\}$. *Thus,* $1 \in X$. *However, for some* $Z \subseteq \mathbb{Z}, X \cap \mathbb{Z} = \{-1\}$. *Thus,* $1 \notin \mathbb{Z}$, *which is a contradiction. Therefore, for all* $X \in P(\mathbb{R})$, $f_5(X) \neq Y$, *so* f_5 *is not surjective.*

6. *f_6 is injective and not surjective. Proof:*

   (a) *Let* $X_1, X_2 \in P(\mathbb{N})$. *Suppose that* $X_1 \neq X_2$ *and let us prove that* $f(X_1) \neq f(X_2)$. *By our assumption, there is* $x \in X_1 \setminus X_2$ *or there is* $x \in X_2 \setminus X_2$. *Since the two cases are symmetric, let us assume without loss of generality that* $x \in X_1 \setminus X_2$. *Then* $2x \in \{2n \mid n \in X_1\}$. *However* $2x \notin \{2n \mid n \in X_2\}$, *just otherwise,* $2x = 2n$ *for some* $n \in X_2$ *which implies that* $x = n \in X_2$, *contradicting the choice of* x.

It follows that $\{2n \mid n \in X_1\} \neq \{2n \mid n \in X_2\}$ *and therefore*

$$f_6(X_1) = \langle\{2n \mid n \in X_1\}, \{2n{+}1 \mid n \in X_1\}\rangle \neq \langle\{2n \mid n \in X_2\}, \{2n{+}1 \mid n \in X_2\}\rangle = f_6(X_2)$$

(b) *Let* $Y = \langle\{0\}, \emptyset\rangle \in P(\mathbb{N}_{even}) \times P(\mathbb{N}_{odd})$. *Suppose towards a contradiction that there is* $X \in P(\mathbb{N})$ *such that*

$$(*) \quad f_6(X) = \langle\{2n \mid n \in X\}, \{2n+1 \mid n \in X\}\rangle = \langle\{0\}, \emptyset\rangle.$$

Then $\{2n \mid n \in X\} = \{0\}$. *It follows that* $0 \in X$ *and therefore* $1 \in \{2n+1 \mid n \in X\}$. *In particular* $\{2n+1 \mid n \in X\} \neq \emptyset$ *contradicting the equality of the pair* $(*)$.

Problem 4. For a function $f : A \to B$ and $C \subseteq A$ define the *pointwise image of* C *by* f as

$$f''C = \{f(c) \mid c \in C\}$$

(a) Prove that if $f : A \to B$ is a function and $C \subseteq A$, then

$$(f''A) \setminus (f''C) \subseteq f''[A \setminus C].$$

(b) Give an example of a function $f : A \to B$ and a subset $C \subseteq A$ such that

$$(f''A) \setminus (f''C) \neq f''[A \setminus C].$$

(c) Prove that if $f : A \to B$ is an injection and $C \subseteq A$, then

$$(f''A) \setminus (f''C) = f''[A \setminus C].$$

Solution. (a) *Let* $b \in f''A \setminus f''C$. *Since* $b \in f''A$, *there is* $a \in A$ *such that* $b = f(a)$. *Since* $b \notin f''C$, $a \notin C$. *It follows that* $a \in A \setminus C$. *We conclude that* $b = f(a) \in f''[A \setminus C]$.

(b) *Let* $f : \{1,2\} \to \{1,2\}$ *defined by* $f(1) = f(2) = 1$*. Let* $A = \{1,2\}$*, and* $C = \{1\}$*. Then*

$$f''\{1,2\} = \{1\},\ f''\{1\} = \{1\} \Rightarrow f''\{1,2\} \setminus f''\{1\} = \emptyset$$

Also

$$\{1,2\} \setminus \{1\} = \{2\} \Rightarrow f''[\{1,2\} \setminus \{1\}] = \{1\}$$

Hence

$$f''\{1,2\} \setminus f''\{1\} = \neq \{1\} = f''[\{1,2\} \setminus \{1\}].$$

(c) *Suppose that* f *is injective and we would like to prove that*

$$(f''A) \setminus (f''C) = f''[A \setminus C].$$

By a double inclusion. In section (a) *we proved* $\subseteq$*. For the other direction, let* $x \in f''[A \setminus C]$*. Then there is* $a \in A \setminus C$ *such that* $f(a) = x$*. By the definition of difference, we would like to prove that* $x \in f''A$ *and* $x \notin f''C$*. Since* $a \in A$*, it follows that* $x = f(a) \in f''A$*. Suppose towards a contradiction that there is* $c \in C$ *such that* $f(c) = x$*. Then* $f(c) = f(a)$*. Since* f *is injective,* $c = a$*. However* $c \in C$ *and* $a \notin C$*, contradiction. Hence* $x \in f''C$*.*

Additional Problems

Problem 5. Let $f : A \to B$ and $g : B \to C$ be function. Prove the following items:

1. If f, g are injective then $g \circ f$ is injective.

2. If f, g are surjective, then $g \circ f$.

Solution. *1. Suppose f, g are injective functions. We want to show that $g \circ f$ is injective. So, let $x_1, x_2 \in A$ such that $(g \circ f)(x_1) = (g \circ f)(x_2)$. By definition, $(g \circ f)(x) = g(f(x))$. So this is equivalent to $g(f(x_1)) = g(f(x_2))$. We want to show that $x_1 = x_2$. By our assumption, g is injective, and thus $f(x_1) = f(x_2)$. Similarly, f is injective, and so we have $x_1 = x_2$ Therefore, $g \circ f$ is injective.*

2. Suppose f, g are surjective functions. We want to show that $g \circ f$ is surjective. So, let $c \in C$. We want to show that there exists $a \in A$ such that $c = (g \circ f)(a)$. Because g is surjective, there exists $b \in B$ such that $c = g(b)$. Similarly, f is surjective, and so there exists $a \in A$ such that $b = f(a)$. Thus, we have $c = g(b) = g(f(a)$, which is equivalent to $c = (g \circ f)(a)$. Therefore, $g \circ f$ is surjective.

Problem 6. Prove that the following functions are invertible and find their inverse:

1. $h : (0, \infty) \to (0, 1)$ defined by $h(x) = \frac{1}{1+x^2}$
2. $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = \begin{cases} n+1 & n \in \mathbb{N}_{even} \\ n-1 & n \in \mathbb{N}_{odd} \end{cases}$.
3. $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by $g(\langle n, m \rangle) = \langle n, n+m \rangle$

Solution. *1. We want to show that h is invertible. This is equivalent to showing that f is both injective and surjective.*

We will first prove that h is injective. Let $x_1, x_2 \in (0, \infty)$ such that $h(x_1) = h(x_2)$. We want to show that $x_1 = x_2$. By definition, then, $\frac{1}{1+{x_1}^2} = \frac{1}{1+{x_2}^2}$.

Standard reduction of this equation implies that ${x_1}^2 = {x_2}^2$. *Since* $x_1, x_2 > 0$, *it follows that* $x_1 = x_2$. *Therefore, h is injective.*

We will now show that h is surjective. Let $y \in (0,1)$. *We will show that there exists some* $x \in (0,\infty)$ *such that* $y = h(x)$. *Equivalently, we want* x *such that* $y = \frac{1}{1+x^2}$. *So, let* $x = \sqrt{\frac{1}{y} - 1}$. *Note that since* $y \in (0,1)$, $\frac{1}{y} - 1 > 0$ *and therefore* x *is a real number in* $(0,\infty)$. *It follows that* $f(x) = \frac{1}{1+\sqrt{\frac{1}{y}-1}^2} = \frac{1}{\frac{1}{y}} = y$. *So h is surjective.*

Therefore, h is invertible.

Inverse: $h^{-1} : (0,1) \longrightarrow (0,\infty)$, $h^{-1}(y) = \sqrt{\frac{1}{y} - 1}$

2. *We will show that* f *is both injective and surjective.*

   We will first prove that f *is injective. Let* $n_1, n_2 \in \mathbb{N}$ *such that* $f(n_1) = f(n_2)$. *We want to prove that* $n_1 = n_2$. *Let us split into cases:*

   (a) *If* $n_1, n_2 \in \mathbb{N}_{even}$, *then* $f(n_1) = n_1 + 1$ *and* $f(n_2) = n_2 + 1$. *It follows that* $n_1 + 1 = n_2 + 1$ *hence* $n_1 = n_2$.

   (b) *If* $n_1, n_2 \in \mathbb{N}_{odd}$, *then* $f(n_1) = n_1 - 1$ *and* $f(n_2) = n_2 - 1$. *It follows that* $n_1 - 1 = n_2 - 1$ *hence* $n_1 = n_2$.

   (c) *If* $n_1 \in \mathbb{N}_{even}$ *and* $n_2 \in \mathbb{N}_{odd}$, *then* $f(n_1) = n_1 + 1$ *is odd and* $f(n_2) = n_2 - 1$ *is even and in particular* $f(n_1) \neq f(n_2)$, *contradicting our assumption. Hence this case is impossible.*

   (d) *The* $n_1 \in \mathbb{N}_{odd}$ *and* $n_2 \in \mathbb{N}_{even}$, *is similar to the one above.*

   We will now prove that f *is surjective. Let* $m \in \mathbb{N}$. *We will show that there is some* $n \in \mathbb{N}$ *such that* $m = f(n)$. *Cases:*

(a) *m is even. Let $n = m + 1$. Then $f(n) = m + 1 - 1 = m$.*

(b) *m is odd. Let $n = m - 1$. Then $f(n) = m - 1 + 1 = m$.*

Therefore, f is surjective, and so f is invertible.

Inverse: $f^{-1} : \mathbb{N} \to \mathbb{N}$, $f^{-1}(m) =$

$$\begin{cases} m+1 & m \in \mathbb{N}_{odd} \\ m-1 & m \in \mathbb{N}_{even} \end{cases}$$

3. *We will first show that g is injective. Let $< n_1, m_1 >, < n_2, m_2 > \in \mathbb{Z} \times \mathbb{Z}$ such that $g(< n_1, m_1 >) = g(< n_2, m_2 >)$. We want to show that $< n_1, m_1 > = < n_2, m_2 >$. Equivalently we need to show that $n_1 = n_2$ and $m_1 = m_2$. By our assumption, we have that $< n_1, n_1 + m_1 > = < n_2, n_2 + m_2 >$. The properties of pairs show us that $n_1 = n_2$. Further, $n_1 + m_1 = n_2 + m_2$, and so $m_1 = m_2$. Thus, $< n_1, m_1 > = < n_2, m_2 >$, and so g is injective.*

   We will now show that g is surjective. Let $< r, s > \in \mathbb{Z} \times \mathbb{Z}$. We want to show that there exists $< n, m > \in \mathbb{Z} \times \mathbb{Z}$ such that $g(< n, m >) = < r, s >$. So let $n = r, m = s - r$. Then $g(< n, m >) = < r, s - r + r > = < r, s >$. Therefore, g is surjective, and thus g is invertible.

   Inverse: $g^{-1} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, $g^{-1}(< r, s >) = < r, s - r >$

Problem 7. Define

$$f_1 : \mathbb{N} \to \mathbb{N} \times \mathbb{N}, \quad f_1(n) = \langle n + 1, n + 2 \rangle$$

$$f_2 : \mathbb{N} \to \mathbb{N}, \quad f_2(n) = n^2$$

$$f_3 : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}, \quad f_3(\langle n, m \rangle) = n - m$$

$$f_4 : \mathbb{N} \to \mathbb{N}, \quad f_4(n) = n + 1$$

Determine if the following compositions are defined and compute them:

1. $f_1 \circ f_2$ and $f_2 \circ f_1$.
2. $f_2 \circ f_2$. and $f_3 \circ f_3$
3. $f_4 \circ f_2$ and $f_2 \circ f_4$.
4. $f_3 \circ f_1 \circ f_2$ and $f_4 \circ f_3 \circ f_2$.

Solution. *1.* $f_1 \circ f_2 : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, $(f_1 \circ f_2)(n) =< n^2 + 1, n^2 + 2 >$

$f_2 \circ f_1$ *is undefined.*

2. $f_2 \circ f_2 : \mathbb{N} \to \mathbb{N}$, $(f_2 \circ f_2)(n) = n^4$

$f_3 \circ f_3$ *is undefined.*

3. $f_4 \circ f_2 : \mathbb{N} \to \mathbb{N}$, $(f_4 \circ f_2)(n) = n^2 + 1$

$f_2 \circ f_4 : \mathbb{N} \to \mathbb{N}$, $(f_2 \circ f_4)(n) = (n + 1)^2$

4. $f_3 \circ f_1 \circ f_2 : \mathbb{N} \to \mathbb{Z}$, $(f_3 \circ f_1 \circ f_2)(n) = -1$

$(f_4 \circ f_3 \circ f_2)(n)$ *is undefined*

Problem 8. Let $A, B \neq \emptyset$ be any set and let $f : A \to B$ be a function. Define a new function using f, as follows, $F : P(A) \to P(B)$ defined by $F(X) = f''X$. Prove that f is invertible if and only if F is invertible.

Solution. *We want to prove that f is invertible if and only if F is invertible. We will prove this by double implication.*

First, suppose f is invertible. We want to prove that F is invertible. We will therefore show that F is a bijection.

1. *We will first show that F is injective. Let $X_1, X_2 \in P(A)$ such that $F(X_1) = F(X_2)$. Equivalently, $\{f(x)|x \in X_1\} = \{f(x)|x \in X_2\}$. We want to show that $X_1 = X_2$. So let $x_1 \in X_1$. We want to show that $x_1 \in X_2$. Denote by $y = f(X_1)$, then by the replacement principle, there exists $y \in F(X_1)$. Since $F(X_1) = F(X_2)$, $y \in F(X_2)$ and therefore, by the replacement principle, there is $x_2 \in X_2$ such that $y = f(x_2)$. We conclude that $f(x_2) = y = f(x_1)$. Since f is injective, $x_1 = x_2$. So, $x_1 \in X_2$ and thus $X_1 \subseteq X_2$. The inclusion $X_2 X_1$ is symmetric. We conclude that $X_1 = X_2$ and therefore, f is injective.*

2. *We will now show that F is surjective. Let $Y \in P(B)$. Then $Y \subseteq B$. We want to show that $Y = F(X)$ for some $X \subseteq A$. Let $X = \{x \in A | f(x) \in Y\}$ and let us prove set equality $F(X) = Y$. Let $y \in Y$, since f is surjective, there exists $x \in A$ such that $f(x) = y$. Since $y \in Y$, $x \in X$ and therefore $y = f(x) \in f''X = f(X)$. For the other direction, let $y \in F(X)$. Then there is $x \in X$ such that $f(x) = y$. By definition of x, $y = f(x) \in Y$. Hence $F(X) = Y$ and therefore F is surjective. So F is a bijection, and therefore F is invertible.*

Therefore, if f is invertible, then F is invertible.

Now suppose that F is invertible. We will show that f is a bijection.

1. *We will first show that f is injective. Let $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$. We want to show that $x_1 = x_2$. Let $X_1 = \{x_1\}, X_2 = \{x_2\}$. Then*

$F(X_1) = \{f(x_1)\}, F(X_2) = \{f(x_2)\}$. As F is injective, it follows that $X_1 = X_2$, and thus $x_1 = x_2$. Therefore, f is injective.

2. *We will now show that f is surjective. Let $y \in B$. We want to show that there exists $x \in A$ such that $y = f(x)$. Let $Y = \{y\}$. Then $Y \in P(B)$. Because F is surjective, there exists X such that $Y = F(X)$. Equivalently, $\{y\} = \{f(x) | x \in X\}$. By the replacement principle and set equality, we have that $y = f(x)$ for some $x \in A$. Therefore, f is surjective. Thus, f is a bijection, and therefore f is invertible.*

Thus, if F is invertible, then f is invertible.

Therefore, f is invertible if and only if F is invertible.