Homework 11

MATH 361 (due December 15) December 9, 2023

We define multiplication of ordinal $\alpha \cdot \beta$ by transfinite induction on β:

- $\alpha \cdot 0 = 0$.
- $\alpha \cdot (\gamma + 1) = (\alpha \cdot \gamma) + \gamma$.
- For limit δ, $\alpha \cdot \delta = \sup_{\gamma<\delta} \alpha \cdot \gamma$.

Note that we use the transfinite definition of '+'.

Problem 1. Prove that $2 \cdot \omega = \omega$ and that $\omega \cdot 2 = \omega + \omega > \omega$.

Problem 2. Prove by transfinite induction that:

1. $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.
2. $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$

Problem 3. Suppose that $\langle A, <_A \rangle, \langle B, <_B \rangle$ are well ordered sets such that $A \cap B = \emptyset$. Recall that the definition of $<_+$ on $A \uplus B$ is $x <_+ y$ if:

- $x, y \in A$ and $x <_A y$. or
- $x, y \in B$ and $x <_B y$. or
- $x \in A$ and $y \in B$.

By HW9, this is a well ordering. Prove that if $otp(A, <_A) = \alpha,\ otp(B, <_B) = \beta$, then $otp(A \uplus B, <_+) = \alpha + \beta$. [Hint: by transfinite induction on β]

Problem 4. Suppose that $\langle A, <_A \rangle, \langle B, <_B \rangle$ are well orders. Recall that the lexicographic order on $A \times B$ is defined as follows:

$$\langle a, b \rangle <_{Lex} \langle a', b' \rangle \text{ iff } a <_A a' \vee (a = a' \wedge b <_B b')$$

By HW9, this is a well ordering. Suppose that $otp(A, <_A) = \alpha$, $otp(B, <_B) = \beta$. Prove that $otp(A \times B, <_{Lex}) = \beta \cdot \alpha$ [Hint: by transfinite induction on α. Use the previous problem.]