Homework 10

MATH 361 | (due Dec 9) | Dec 2, 2022

Problem 1. Suppose that $\langle A, <_A \rangle$ is a well-ordered set. Prove that if $f : A \to A$ is order-preserving then $f = id_A$.

Problem 2. Prove that if A is countable the A can be well-ordered.

Instruction: Split into two cases- first prove that every linear strong order on a finite set is a well order. If A is infinitely countable, then by taking any bijection $f : \mathbb{N} \to A$, we can define $<_A$ on A by $a <_A b$ if and only if $f^{-1}(a) < f^{-1}(b)$. Prove that $\langle A, <_A \rangle \simeq \langle \mathbb{N}, < \rangle$ and deduce that $\langle A, <_A \rangle$ is a well ordered set.

Problem 3. Prove that if $\langle A, <_A \rangle$ is a well-ordered set and $X \subseteq A$ is an initial segment (i.e. $\forall x \in X \forall a \in A,\ a <_A x \Rightarrow a \in X$) then either $X = A$ or $\exists a \in A$ such that $X = A_{<A}[a]$.

Hint: If $X \neq A$ let $a = \min_{<_A}(A \setminus X)$ (why does it exists?), prove that $X = A_{<A}[a]$.

Problem 4. Prove that the axiom of foundation implies that there is no x such that $x \in x$.

Problem 5. Prove that if A is a set of ordinals then $\bigcup A$ is an ordinal and moreover $\bigcup A = \sup(A)$ i.e.:

1. $\bigcup A$ is an upper bound for A, namely, for every $\alpha \in A$, $\alpha \leq \bigcup(A)$.

2. If $\beta \in On$ is an upper bound for A then $\beta \geq \bigcup A$.

Additional problems

Problem 6. Suppose that $\langle A, <_A \rangle, \langle B, <_B \rangle$ are well ordered sets such that $A \cap B = \emptyset$. Define $<_+$ on $A \uplus B$ by $x <_+ y$ if:

- $x, y \in A$ and $x <_A y$. or
- $x, y \in B$ and $x <_B y$. or
- $x \in A$ and $y \in B$.

Prove that $<_+$ is a well ordering of $A \uplus B$.

Problem 7. Suppose that $\langle A, <_A \rangle, \langle B, <_B \rangle$ are well orders. Define the lexicographic order on $A \times B$ as follows:

$$\langle a, b \rangle <_{Lex} \langle a', b' \rangle \text{ iff } a <_A a' \vee (a = a' \wedge b <_B b')$$

Prove that $\langle A \times B, <_{Lex} \rangle$ is a well ordering.

Problem 8. Prove that if α is an ordinal then $\alpha \cup \{\alpha\}$ is an prdinal.

Problem 9. Prove that if $C \neq \emptyset$ is a set of ordinals then $\bigcap C$ is an ordinal and $\bigcap C = \min_{\in}(C)$.

Problem 10. Prove that if X is transitive than $P(X)$ is transitive.