# Homework 1

MATH 361 (due September 2) August 2, 2022

**Problem 1.** Prove or disprove the following items:

(a) $\{1, -1\} \subseteq \mathbb{Z}$.

(b) $7 \in \{n \in \mathbb{N} \mid |n^2 - n - 3| \leq 5\}$.

(c) $27 \in \{n^2 - n - 3 \mid n \in \mathbb{N}\}$.

(d) $-3 \in \{n^2 - 3 \mid n \in \mathbb{N}_+\}$.

(e) $\{1, -1\} \in \{X \subseteq \mathbb{Z} \mid 2 \in X\}$.

(f) $\{r \in \mathbb{R} \mid \exists q \in \mathbb{Q}. r + q \in \mathbb{Q}\} = \mathbb{Q}$.

(g) $\{-1, 0, 1\} \subseteq \{x \in \mathbb{N} \mid x^2 = |x|\}$. (Here $|x|$ is the absolute valure of the real number $x$)

(h) $\{x \in \mathbb{R} \mid \{x, x+1\} \subseteq [0, 2)\} \subseteq [0, 1]$.

(i) $\mathbb{Q} \subseteq \{x \in \mathbb{R} \mid |\{x, x + \sqrt{2}\} \cap \mathbb{Q}| = 1\}$

**Solution.** (a) Prove! $1 \in \mathbb{Z}$ and $-1 \in \mathbb{Z}$ therefore $\{1, -1\} \subseteq \mathbb{Z}$.

(b) Disprove! $|7^2 - 7 - 3 = 39| = 39 > 5$ so by the comprehension principle 7 is not in the set.

(c) Prove! By the replacement principle, we need to prove that there if $n \in \mathbb{N}$ such that $27 = n^2 - n - 3$. Let $n = 6$, then $6^2 - 6 - 3 = 36 - 9 = 27$.

(d) Disprove. Suppose toward a contradiction that $-3 \in \{n^2 - 3 \mid n \in \mathbb{N}_+\}$, then by the replacment principle there is $n \in \mathbb{N}_+$ such that $-3 = n^2 - 3$. Hence $n^2 = 0$ which implies that $n = 0$. However $n \in \mathbb{N}_+$, contradiction.

(e) Disprove! $2 \notin \{1, -1\}$ and therefore by the comprehension principle $\{1, -1\}$ is not in the set.

(f) Prove! by double inclusion:

$\underline{\subseteq}$: Let $x \in \{r \in \mathbb{R} \mid \exists q \in \mathbb{Q},\ r + q \in \mathbb{Q}\}$. There by the comprehension principle, there is $p \in \mathbb{Q}$ such that $r + p \in \mathbb{Q}$. It follows that $r = (r + p) - p$ is the difference of two rational numbers and therefore $r \in \mathbb{Q}$.

$\underline{\supseteq}$: Let $r \in \mathbb{Q}$, then $0 \in \mathbb{Q}$ is such that $r + 0 = r \in \mathbb{Q}$. It follows that there is $q \in \mathbb{Q}$ such that $r + q \in \mathbb{Q}$. By the comprehension principle, $r \in \{r \in \mathbb{R} \mid \exists q \in \mathbb{Q},\ r + q \in \mathbb{Q}\}$.

(g) Disprove! $-1 \in \{1, 0, -1\}$ but $-1 \notin \mathbb{N}$ hence by the comprehension principle $-1 \notin \{x \in \mathbb{N} \mid x^2 = |x|\}$. It follows that $\{1, 0, -1\} \nsubseteq \{x \in \mathbb{N} \mid x^2 = |x|\}$.

(h) Prove! Let $r \in \{x \in \mathbb{R} \mid \{x, x + 1\} \subseteq [0, 2)\}$ then $r, r + 1 \in [0, 2)$ and therefore $0 \leq r$ and $r + 1 < 2$. It follows that $r < 1$ and therefore $r \in [0, 1)$.

(i) Per demand.

**Problem 2.** Prove that if $A, B, C$ are sets then

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

**Solution.** We prove a double inclusion:

$\underline{\subseteq}$: Let $x \in A \cup (B \cap C)$. The by the definition of union either $x \in A$ or $x \in B \cap C$. We split into cases:

(a) If $x \in A$ then $x \in A \cup B$ and $x \in A \cup C$ by the definition of union. By definition of intersection $x \in (A \cup B) \cap (AC)$.

(b) If $x \in B \cap C$, then by definition on intersection $x \in B$ and $x \in C$. Hence $x \in A \cup B$ and $x \in A \cup C$ by the definition of union and again by definition of intersection $x \in (A \cup B) \cap (A \cap C)$.

In any case $x \in (A \cup B) \cap (A \cup C)$.

$\supseteq$: Let $x \in (A \cup B) \cap (A \cup C)$. Then by definition of intersection $x \in A \cup B$ and $x \in A \cup C$. LEt us split into cases:

(a) If $x \in A$, then by definition of union $x \in A \cup (B \cap C)$.

(b) If $x \notin A$, since $x \in A \cup B$ and $x \in A \cup C$, then by definition of union $x \in B$ and $x \in C$. By definition of intersection $x \in B \cap C$. By definition of union $x \in A \cup (B \cap C)$.

**Problem 3.** Let $\mathcal{B}$ be a nonempty set of sets and let $A$ be any set. Show that

(a) $A \cap \bigcup \mathcal{B} = \bigcup \{A \cap B \mid B \in \mathcal{B}\}$.

(b) $A \setminus \bigcap \mathcal{B} = \bigcup \{A \setminus B \mid B \in \mathcal{B}\}$.

**Solution.** We will prove item (1) as an example: By double inclusion:

$\subseteq$: Let $x \in A \cap \bigcup \mathcal{B}$. By definition of intersection $x \in A$ and $x \in \bigcup \mathcal{B}$. By definition of generalized union, there is $B_0 \in \mathcal{B}$ such that $x \in B_0$. It follows that $x \in A \cap B_0$. Since $A \cap B_0 \in \{A \cap B \mid B \in \mathcal{B}\}$, and by the definition of generalized union, $x \in \bigcup \{A \cap B \mid B \in \mathcal{B}\}$.

$\underline{\supseteq}$: Let $x \in \bigcup\{A \cap B \mid B \in \mathcal{B}\}$. Then by the definition of generalized union, there is $B \in \mathcal{B}$ such that $x \in A \cap B$. By definition of intersection $x \in A$ and $x \in B$. It follows the $x \in \bigcup \mathcal{B}$ and by definition on intersection $x \in A \cap \bigcup \mathcal{B}$.

**Problem 4.** Let $A, B$ be sets. prove that for any $a \in A$ and $b \in B$, $\langle a, b\rangle \in P(P(A \cup B))$. Conclude formally from the axioms that there is a unique set $D$ which equals $A \times B$. Namely, that for every $x$, $x \in D$ if and only if $x = \langle a, b\rangle$ for some $a \in A$ and $b \in B$.

**Problem 5.** Prove that for every sets $A, B, C$,

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

**Solution.** Let $A, B, C$ be any sets. We want to prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$. We will prove this by double inclusion.

1. We will first show that $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$. Let $(x, y) \in A \times (B \cap C)$. We will show that $(x, y) \in (A \times B) \cap (A \times C)$. By definition, $x \in A$ and $y \in (B \cap C)$. Thus, $y \in B$, and so $(x, y) \in A \times B$. Similarly, $y \in C$ and thus $(x, y) \in A \times C$. Therefore, $(x, y) \in (A \times B) \cap (A \times C)$.

2. We will now show that $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$. Let $(x, y) \in (A \times B) \cap (A \times C)$. Then $(x, y) \in A \times B$ and $(x, y) \in A \times C$. By definition, $x \in A$ and $y \in B$. Similarly, $x \in A$ and $y \in C$. It follows that $x \in A$ and $y \in B \cap C$. Then by definition, $(x, y) \in A \times (B \cap C)$.

Therefore, $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

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## Additional problems:

**Problem 6.** Prove implications (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) of Proposition 2.9.

**Problem 7.** Let $X$ and $Y$ be sets.

(i) Prove that $Y \setminus (Y \setminus X) = X \cap Y$.

(ii) Prove that $X \subseteq Y$ if and only if $X \cup Y = Y$.

(iii) Deduce that $X \subseteq Y$ if and only if $Y \setminus (Y \setminus X) = X$.

**Solution.** (i) Let $X$ and $Y$ be sets. We want to prove that $Y \backslash (Y \backslash X) = X \cap Y$. We will prove this by double inclusion.

(1) We will first prove that $Y \backslash (Y \backslash X) \subseteq X \cap Y$. Let $x \in Y \backslash (Y \backslash X)$. We want to prove $x \in X \cap Y$. By the separation principle, $x \in Y \wedge x \notin (Y \backslash X)$. Using the definition of $Y - X$ and the separation principle, we have:

$$\begin{aligned} &\neg(x \in Y \wedge x \notin X) \\ \equiv\ &\neg(x \in Y) \vee \neg(x \notin X) \\ \equiv\ &x \notin Y \vee x \in X \end{aligned}$$

As $x \in Y$, we must have $x \in X$. Thus, $x \in Y \wedge x \in X$, and therefore $x \in X \cap Y$.

(2) We will now show that $X \cap Y \subseteq Y - (Y - X)$. Let $x \in X \cap Y$. Then $x \in X \wedge x \in Y$. We want to show that $x \in Y - (Y - X)$. As $x \in Y$ is given, we must show that $x \notin Y - X$. By the previous result, this is equivalent to $x \notin Y \vee x \in X$. By our assumption,

$x \in Y$, so we must have $x \in X$. Therefore, $x \notin Y - X$. Thus, by definition, $x \in Y - (Y - X)$, and so $X \cap Y \subseteq Y - (Y - X)$.

Therefore, by double inclusion, $Y \backslash (Y \backslash X) = X \cap Y$.

(ii) Let $X, Y$ be sets. We want to prove that $X \subseteq Y \iff X \cup Y = Y$. We will prove this via double implication.

(1) We will first show that $X \subseteq Y \implies X \cup Y = Y$. Assume that $X \subseteq Y$. We want to prove that $X \cup Y = Y$. We will prove this by double inclusion.

i. Clearly, $Y \subseteq X \cup Y$.

ii. So, we will now show that $X \cup Y \subseteq Y$. Let $x \in X \cup Y$. Then $x \in X \vee x \in Y$. If $x \in Y$, we are done. If $x \in X$, then by our assumption, $x \in Y$. Thus, $X \cup Y \subseteq Y$.

Thus, by double inclusion, $X \cup Y = Y$, and so $X \subseteq Y \implies X \cup Y = Y$.

(2) We will now show that $X \cup Y = Y \implies X \subseteq Y$. Assume that $X \cup Y = Y$, and $x \in X$. We want to show that $x \in Y$. Clearly, $x \in X \cup Y$. By our assumption, we know $X \cup Y = Y$. Then by definition, $x \in Y$. Thus, $X \cup Y = Y \implies X \subseteq Y$.

Therefore, by double implication, $X \subseteq Y \iff X \cup Y = Y$.

(iii) It follows from these results that $X \subseteq Y \iff Y - (Y - X) = X$. We will show this by double implication:

(1) We will first show that $X \subseteq Y \implies Y - (Y - X) = X$. Assume that $X \subseteq Y$. By (i), we know that $Y - (Y - X) = X \cap Y$. By (ii) and our assumption, we know $X \cup Y = Y$. Thus, substitution and transitivity gives us

$$(X \subseteq Y \implies Y - (Y - X) = X) \equiv (X \cup Y = Y \implies X \cap Y = X)$$

This implication was proven previously, so we have $X \subseteq Y \implies Y - (Y - X) = X$.

(2) We will now show that $Y - (Y - X) = X \implies X \subseteq Y$. Assume that $Y - (Y - X) = X$. Similarly to (1), we know from (i) that $Y - (Y - X) = X \cap Y$. Then by substitution, $X \cap Y = X$, which is known to imply $X \subseteq Y$. Thus $Y - (Y - X) = X \implies X \subseteq Y$.

Therefore, by double implication, $X \subseteq Y \iff Y - (Y - X) = X$.

**Problem 8.** Compute the following sets. No proof required.

1. $\Big\{a + b : a \in \{0, 5\}, b \in \{2, 4\}\Big\} \setminus \{7, 10\}$.
2. $(1, 3) \cup [2, 4)$
3. $\mathbb{Z} \cap [0, \infty)$
4. $\mathbb{N}_{even} \Delta \mathbb{N}_{+}$

**Solution.** 1) $\{2, 4, 9\}$

2) $(1, 4)$

3) $\mathbb{N}$

4) $\{0\} \cup \mathbb{N}_{odd}$

**Problem 9.** Prove that for every two sets $A, B$ the following are equivalent:

- $A \subseteq B$.
- $P(A \cup B) = P(B)$.
- $P(A) \subseteq P(B)$.

[Remember: You are allowed to use the propositions and statements which appear in the class notes.]

**Solution.** We will show these equivalences via a chain of implications..

1. We will first show that $A \subseteq B \implies P(A \cup B) = P(B)$. Let $A, B$ be sets such that $A \subseteq B$. We want to prove $P(A \cup B) = P(B)$. We will prove this by double inclusion.

   (a) We will first show that $P(A \cup B) \subseteq P(B)$. Let $X \in P(A \cup B)$. Then $X \subseteq A \cup B$. So, let $x \in X$. By definition, $x \in A \vee x \in X$. Then we have the following cases:

      i. Suppose $x \in B$. Then $X \subseteq B$, and thus $X \in P(B)$.

      ii. Suppose $x \in A$. Then by our assumption, $x \in B$ and so we know $X \in P(B)$.

   (b) We will now show that $P(B) \subseteq P(A \cup B)$. Let $X \in P(B)$ and $x \in X$. Then $X \subseteq B$ and so $x \in B$. We want to show that $X \in P(A \cup B)$. We know $x \in B$, so clearly $x \in A \cup B$. Thus, $X \subseteq A \cup B$ and so $X \in P(A \cup B)$.

Therefore, $A \subseteq B \implies P(A \cup B) = P(B)$.

2. We will now show that $P(A \cup B) = P(B) \implies P(A) \subseteq P(B)$. Assume $P(A \cup B) = P(B)$, and let $X \in P(A)$ and $x \in X$. Then $x \in A$, and so clearly $x \in A \cup B$. Thus, $X \subseteq A \cup B$ and $X \in P(A \cup B)$. Therefore, by our assumption, we have $X \in P(B)$.

3. We will finally show that $P(A) \subseteq P(B) \implies A \subseteq B$. So, assume $P(A) \subseteq P(B)$ and let $a \in A$. Then there exists some $X \subseteq A$ such that $a \in X$. Then $X \in P(A)$, so by our assumption $X \in P(B)$. Therefore, $X \subseteq B$ and thus $a \in B$.

Therefore, these statements are equivalent.