Math 300 Intro Math Reasoning
Worksheet 9: Equinumerability

(1) Prove that $P(\mathbb{N} \times \mathbb{Z}) \sim {}^{\mathbb{N}}\{0,1\}$.

Solution By a theorem from class $\mathbb{Z} \sim \mathbb{N}$ and therefore by a theorem from class (which is also the next problem) if $A \sim A'$ and $B \sim B'$ then $A \times B \sim A' \times B'$. In particular $\mathbb{N} \times \mathbb{N} \sim \mathbb{N} \times \mathbb{Z}$. Another theorem from class is used to show $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$ and another one, to show that if $A \sim B$ then $P(A) \sim P(B)$. So we conclude that since $\mathbb{N} \times \mathbb{Z} \sim \mathbb{N}$, $P(\mathbb{N} \times \mathbb{Z}) \sim P(\mathbb{N})$. Finally, we prove that for every set A, ${}^{A}\{0,1\} \sim P(A)$ and therefore $P(\mathbb{N} \times \mathbb{Z}) \sim P(\mathbb{N}) \sim {}^{\mathbb{N}}\{0,1\}$.

(2) Suppose that $A \sim A'$ and $B \sim B'$. Prove that $A \times B \sim A' \times B'$

Solution. Let $f : A \to B$ and $g : C \to D$ be bijections. Define $h : A \times C \to B \times D$ $h(\langle a, c\rangle) = \langle f(a), g(c)\rangle$. Prove that h is one-to-one. Let us prove for example that h is onto. Let $\langle b, d\rangle \in B \times D$. Since f, g are onto, there are $a \in A$ and $c \in C$ such that $f(a) = b$ and $g(c) = d$. Then $\langle a, c\rangle \in A \times C$ and $h(\langle a, c\rangle) = \langle f(a), g(c)\rangle = \langle b, d\rangle$.