Math 300 Intro Math Reasoning
Worksheet 05: Set Theory

We define for every $A \subseteq \mathbb{R}$ and $r \in \mathbb{R}$

$$A + r = \{a + r \mid a \in A\}$$

(1) Compute (not proof):

(1) $\{1, 5\} + 0.5 = \{1.5, 5.5\}$.
(2) $\mathbb{N} + 1 = \mathbb{N}_+$.
(3) $\mathbb{Z} + 1 = \mathbb{Z}$.
(4) $\emptyset + r = \emptyset$.

(2) Prove or disprove:

(1) If $A \subseteq B$ then $A + r \subseteq B + r$.
Solution. Suppose that $A \subseteq B$. WTP $A + r \subseteq B + r$. Let $x \in A + r$. WTP $x \in B + r$. By the replacement principle, there is $a \in A$ such that $x = a + r$. Since $A \subseteq B$, $a \in B$. Therefore there is $b \in B$ such that $x = b + r$ which again by replacement implies that $x \in B + r$.
(2) If for some $r, s \in \mathbb{R}$, $A + r \subseteq B + s$ then $A \subseteq B$.
Solution. This is false, for every $A = \{1\}$, $B = \{2\}$ then $\{1\} + 1 \subseteq \{2\} + 0$, however $\{1\} \nsubseteq \{2\}$.
(3) $A + 0 = A$
Solution. Let us prove this set equality by a double inclusion.
$\supseteq$: Let $a \in A$. WTP $a \in A + 0$. We have that $a = a + 0$, and therefore there is $x \in A$ such that $x + 0 = a$. By replacement, $a \in A + 0$.
$\subseteq$ For the other direction, suppose that $x \in A + 0$, the by replacement, there is $a \in A$ such that $x = a + 0 = a$ and therefore $X \in A$.
Since we proved a double inclusion it follows that $A + 0 = A$

(3) Prove that for every $r \in \mathbb{R}$, $\mathbb{Q} + r = \mathbb{Q}$ if and only if $r \in \mathbb{Q}$.

Solution. Let $r \in \mathbb{R}$. Let us prove the equivalence by a double implication:

$\Rightarrow$ Suppose that $\mathbb{Q}+r = \mathbb{Q}$. WTR $r \in \mathbb{Q}$. Note that by replacement, $r = 0+r \in \mathbb{Q}+r$. By the set equality assumption $r \in \mathbb{Q}$.
$\Leftarrow$ Suppose that $r \in \mathbb{Q}$ and let us prove that $\mathbb{Q} + r = \mathbb{Q}$ by a double inclusion.
$\subseteq$ Let $x \in \mathbb{Q} + r$. WTP $x \in \mathbb{Q}$. By replacement, there is $q \in \mathbb{Q}$ such that $x = q + r$. Since both q and r are rationals, $x \in \mathbb{Q}$.
$\supseteq$ Let $x \in \mathbb{Q}$ WTP $x \in \mathbb{Q}+r$. Define $q = x-r$, the again since x, r are rationals, $q \in \mathbb{Q}$. Also note that $x = q + r$ and therefore $x \in \mathbb{Q} + r$.
Since we proved a double inclusion it follows that $\mathbb{Q} = \mathbb{Q} + r$,