Math 300 Intro Math Reasoning
Worksheet 03: Mathematical logic

(1) Prove the following statement: An integer is divisible by 5 if and only if its last digit is divisible by 5.

[Hint: To formally refer to the unit number of an integer n, decompose $n = 10k+d$ where k is some integer and $0 \leq d \leq 9$. Then d is the unit digit of n.]

Solution Let n be an integer and suppose that $n = 10k+d$ where $k, d \in \mathbb{Z}$ and $0 \leq d \leq 9$. Let us prove the statement by a double implication.

- $\Rightarrow$: If 5 divides n, then $n = 5l$ for some $l \in \mathbb{Z}$ and therefore $d = n - 10k = 5l - 10k = 5(l - 2k)$. Since $l - 2k \in \mathbb{Z}$, 5 divides d.
- $\Leftarrow$: If 5 divides d, then $d = 5l$ for some $l \in \mathbb{Z}$. Then $n = 10k + 5l = 5(2k + l)$ and since $2k + l \in \mathbb{Z}$, 5 divides n.

(2) Prove that for all integers n and m, if n is multiple of 6 or m is multiple of 9 then n^2m is a multiple of 9.

Solution: Suppose that n is a multiple of 6 or m is a multiple of 9. WTP n^2m is a multiple of 9. Let us split into cases:

(1) If 6 divides n, then $n = 6k$ for some $k \in \mathbb{Z}$ and therefor $n^2m = (6k)^2m = 9(4k^2m)$. Since $4k^2m \in \mathbb{Z}$, 9 divides n^2m.
(2) If 9 divides m, then since m divides n^2m, it follows from what we proved in class that 9 divides n^2m.

In any case 9 divides n^2m. (3) Let a, b be integers with $b \neq 0$. Prove that any integer solution to the quadratic equation $x^2 + ax + b = 0$ divides b.

Solution. Suppose z is an integer solution to $x^2 + ax + b = 0$. WTP z divides b. By our assumption, $z^2 + za + b = 0$, hence $b = -z^2 - za = z(-z - a)$. Since $-z - a \in \mathbb{Z}$, it follows that z divides b.