Homework 6

MATH 300 | (due March 22) | March 11, 2021

Problem 1. Describe the set $P(\{\emptyset, \{\emptyset\}\})$ using the list principle. No proof required.

Solution $P(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

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Problem 2. Prove that for every two sets A, B the following are equivalent:

1. $A \subseteq B$.

2. $P(A \cup B) = P(B)$.

3. $P(A) \subseteq P(B)$.

Solution Let us prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) Suppose that $A \subseteq B$. WTP $P(A \cup B) = P(B)$. By a proposition in class, $A \subseteq B$ implies $A \cup B = B$ and therefore $P(A \cup B) = P(B)$.

(2) $\Rightarrow$ (3) Suppose that $P(A \cup B) = P(B)$. WTP $P(A) \subseteq P(B)$. Let $X \in P(A)$, then by definition of powerset $X \subseteq A$. Since $X \subseteq A$ and $A \subseteq A \cup B$, it follows that $X \subseteq A \cup B$. By definition of powerset $X \in P(A \cup B)$ and since $P(A \cup B) = P(B)$, $X \in P(B)$.

(3) $\Rightarrow$ (1) Suppose that $P(A) \subseteq P(B)$. WTP $A \subseteq B$. (you can prove it directly, but here is a simpler proof) Since $A \subseteq A$, $A \in P(A)$ and since $P(A) \subseteq P(B)$, $A \in P(B)$. By definition of powerset, this means that $A \subseteq B$, as wanted.

Problem 3. Prove that for every natural number n, the number $4^n - 1$ is multiple of 3.

Solution

Base $n = 0, 4^0 - 1 = 0$ and $0 = 3 \cdot 0$.

I.H Suppose that $4^n - 1$ is a multiple of 3.

Induction step WTP $4^{n+1} - 1$ is a multiple of 3.

$$4^{n+1} \cdot 1 = 4 \cdot 4^n - 1 = 4(4^n - 1) + 3$$

Since both $3, 4^n - 1$ are multiples of 3 by the I.H, also $4(4^n - 1) + 3$ is a multiple of 3 and therefore $4^{n+1} - 1$ is a multiple of 3.

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Problem 4. Prove that for every natural number n, the number $17n^3+103n$ is multiple of 6.

Solution Note that $17n^3 + 103n = 120n + 17n(n^2 - 1)$. Since $120n$ is divisible by 6 is remains to prove that $17n(n^2-1)$ is divisible by 6. This can either be proven by induction of just note that $17n(n^2-1) = 17(n-1)n(n+1)$ and at least on of $(n-1)n(n+1)$ is even (i.e. divisible by 2 and at least one is divisible by 3, so the product is divisible by 6.

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Problem 5. Prove that for every natural number n,

$$1 + 4 + 4^2 + \cdots + 4^n = \frac{4^{n+1} - 1}{3}.$$

Solution

Base For $n = 0$ we have $1 = \frac{4^{0+1}-1}{3} = \frac{3}{3}$

I.H Suppose that

$$1 + 4 + 4^2 + \cdots + 4^n = \frac{4^{n+1} - 1}{3}.$$

Induction step WTP

$$1 + 4 + 4^2 + \cdots + 4^n + 4^{n+1} = \frac{4^{n+2} - 1}{3}.$$

indeed by the induction hypothesis

$$1+4+4^2+\cdots+4^n+4^{n+1} = \frac{4^{n+1} - 1}{3}+4^{n+1} = \frac{4^{n+1} - 1 + 3 \cdot 4^{n+1}}{3} = \frac{4 \cdot 4^{n+1} - 1}{3} = \frac{4^{n+2} - 1}{3}.$$

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Problem 6. Prove that for all odd natural numbers n, the number $2^n + 1$ is multiple of 3.

[Hint: First find an equivalent statement about all positive integers. Then use induction.]

Solution Since any odd number as the form $2k + 1$, it suffices and $2^{2k+1} + 1 = 2 \cdot 4^k + 1$, it suffices to prove that for every natural number k, $2 \cdot 4^k + 1$ is a multiple of 3. By induction on k

Base For $k = 0$ we have $2 \cdot 4^0 + 1 = 3$.

I.H Suppose that $2 \cdot 4^k + 1$ is a multiple of 3.

Induction step Let us prove that $2 \cdot 4^{k+1} + 1$ is a multiple of 3. Indeed $2 \cdot 4^{k+1} + 1 = 8 \cdot 4^k + 1 = 6 \cdot 4^k + 2 \cdot 4^k + 1$. Clearly $6 \cdot 4^k$ is a multiple of 3 and $2 \cdot 4^k + 1$ is a multiple of 3 by I.H. Therefore $2 \cdot 4^{k+1} + 1$ is a multiple of 3.

Next problem is optional and will not be graded for points.

Problem 7. Criticize the following obviously wrong argument:

> «All horses are the same color. Specifically, every finite set of horses is monochromatic.»
>
> *Proof.* We argue by induction. The statement is clearly true for sets of size 1. Assume by induction that all sets of n horses are monochromatic, and consider a set of size $n + 1$. The first n horses are all the same color. The last n horses are all the same color. Because of the overlap, this means that all $n + 1$ horses are the same color. So by induction, all finite sets of horses are all the same color, and so all horses are the same color. □

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Next problem is optional.

Problem 8. Suppose $x \geq -1$. Prove that for all natural number n,

$$(1 + x)^n \geq 1 + nx.$$

Next problem is optional.

Problem 9. Suppose $x \geq -1$. Prove that for all natural number n,

$$(1 + x)^n \geq 1 + nx.$$