Mathematical Reasoning-Midterm 1

MATH 300 (Instructor: Tom Benhamou) Oct 18, 2024

Instruction

The midterm consists of 3 problems, each worth 34 points (The maximal grade is 100). For this you will have one hour. The identities file will be appended to the exam and no other material is allowed. The answers to the problems should be answered in the designated areas.

Problems

Problem 1. For each of the following statements determine if it is true or false. No explanation is required, circle the correct answer:

a. $-5 \in \left\{m \in \mathbb{Z} \mid m^2 \notin \{1, 4, 25, 100\}\right\}$. True \ <u>False</u>

b. $\alpha \Rightarrow (\beta \Rightarrow \alpha)$ is a tautology. <u>True</u> \ False

c. $A \times (B \cap C) = (A \times B) \cap (A \times C)$. <u>True</u> \ False

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Problem 2. Prove or disprove. If you choose to disprove, give a counter example for A, B without further explanation:

(1) For every two sets A, B, $P(A \cap B) = P(A) \cap P(B)$.

(2) For every two sets A, B, $P(A \cup B) = P(A) \cup P(B)$.

Solution: For (1) we prove that $P(A \cap B) = P(A) \cap P(B)$ by a double inclusion.

$\underline{P(A \cap B) \subseteq P(A) \cap P(B)}$: Let $X \in P(A \cap B)$ WTP $X \in P(A) \cap P(B)$. By definition of powerset, $X \subseteq A \cap B$. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, it follows that $X \subseteq A$ and $X \subseteq B$. Hence $X \in P(A)$ and $X \in P(B)$ and by definition of intersection $X \in P(A) \cap P(B)$.

$\underline{P(A) \cap P(B) \subseteq P(A \cap B)}$: Let $X \in P(A) \cap P(B)$, WTP $X \in P(A \cap B)$. By assumption $X \in P(A)$ and $X \in P(B)$, and therefore $X \subseteq A$ and $X \subseteq B$. To see that $X \subseteq A \cap B$, let $x \in X$, the $x \in A$ and $x \in B$ (since $X \subseteq A$ and $X \subseteq B$) and therefore $x \in A \cap B$. It follows that $X \subseteq A \cap B$, namely $X \in P(A \cap B)$.

For (2) we disprove, take $A = \{1\}$ and $B = \{2\}$. Let us explain although it was not requires in the exam. Then $\{1, 2\} \in P(\{1, 2\}) = P(A \cup B)$ and $\{1, 2\} \notin P(\{1\}) = P(A)$ and $\{1, 2\} \notin P(\{2\}) = P(B)$, hence $\{1, 2\} \notin P(A) \cup P(B)$. It follows that $P(A \cup B) \neq P(A) \cup P(B)$.

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Problem 3. Prove by induction that for every n, $2^{n+1}+5^n$ is divisible by 3.

Solution: For the base case, $2^{0+1}+5^0=2+1=3$ is divisible by 3.

The induction hypothesis: Suppose that $2^{n+1}+5^n$ is divisible by 3

Induction step, let us prove that $2^{n+2}+5^{n+1}$ is divisible by 3. By the induction hypothesis there is l such that $2^{n+1}+5^n=3l$. Now for $n+1$ we have,

$$2^{n+2}+5^{n+1}=2(2^{n+1})+5(5^n)=2(2^{n+1}+5^n)+3(5^n)=2(3l)+3(5^n)=3(2l+5^n).$$

Define $k=2l+5^n$, then k is an integer. Since we have shown that $2^{n+2}+5^{n+1}=3k$, $2^{n+2}+5^{n+1}$ is divisible by 3

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