Midterm 1- Solutions

MATH 300 (Instructor: Tom Benhamou)

Instruction

The midterm consists of 3 problems, each worth 34 points (The maximal grade is 100). For this you will have 45 minutes during class. The identities file will be appended to the exam and no other material is allowed. The answers to the problems should be answered in the designated areas.

Solutions

Problem 1. For each of the following statements determine if it is true are false. No explanation required:

a. $1 \in \Big\{-1, 2, \{1, 1\}, \{\{1\}\}\Big\}$ (False)

b. $13 \in \Big\{n^2 + n + 1 \mid n \in \mathbb{N}_{even}\Big\}$. (False- note that n must be even)

c. $6 \in \Big\{n \in \mathbb{N} \mid \{m \in \mathbb{Z} \mid m^2 + n \leq 5\} = \emptyset\Big\}$ (True)

Problem 2. Prove that for all positive integers a, b, if both $2a + b$ and $a - 2b$ are divisible by 3 then a is divisible by 3.

Proof. Let a, b be any positive integers and suppose that $2a + b$ and $a - 2b$ are divisible by 3. We need to prove that a is divisible by 3. By assumption, there are integers k, l such that $2a + b = 3k$ and $a - 2b = 3l$. Reduce the two equations:

$$3k - 3l = (2a + b) - (a - 2b) = a + 3b$$

Hence

$$a = 3k - 3l - 3b = 3(k - l - b)$$

Since $k-l-b$ is an integers it follows that a is divisible by 3. $\square$

Problem 3. Let α be the statement:

$$\forall x \in \mathbb{N}.\Big(\exists y \in \mathbb{N}.\big((y < x) \vee (y^2 < x^2)\big)\Big)$$

Solution:

$$\neg\alpha \equiv \underline{\exists x \in \mathbb{N}.\Big(\forall y \in \mathbb{N}.\big((y \geq x) \wedge (y^2 \geq x^2)\big)\Big)}$$

b. Prove or disprove α.

solution: Disprove α. Let us prove $\neg\alpha \equiv \exists x \in \mathbb{N}.\Big(\forall y \in \mathbb{N}.\big((y \geq x) \wedge (y^2 \geq x^2)\big)\Big)$, so we need to prove an existential statement. Define $x = 0$, we need to prove that $\forall y \in \mathbb{N}.(y \geq 0) \wedge (y^2 \geq 0)$ (note that this is universal). Let $y_0 \in \mathbb{N}$, we need to prove that $(y_0 \geq 0) \wedge (y_0^2 \geq 0)$. Indeed, since y_0 is a natural number then $y_0 \geq 0$ and therefore also $y_0^2 \geq 0$.