Math 300
Midterm Examples

February 25, 2024

Instruction

The midterm consists of 3 problems, each worth 34 points (The maximal grade is 100). For this you will have 45 minutes during class. The identities file will be appended to the exam and no other material is allowed. The answers to the problems should be answered in the designated areas.

Examples for problems

Problem 1. Determine whether “$(A \vee (A \Rightarrow B)) \Rightarrow (A \wedge B)$" is a tautology. Justify your answer using truth tables:

solution This is a tautology.

Problem 2. Determine whether the conclusion logically follows from the premises:

Premise 1: $A \wedge (B \Rightarrow C)$

Premise 2: $A \vee B$

Conclusion: $\neg C$

Solution. The conclusion does not follow. For example, set $V(C) = T$ $V(B) = F$ and $V(A) = T$. Then $V(B \Rightarrow C) = T$ and therefore $V(A \wedge (B \Rightarrow C)) = T$. Also $V(A \vee B) = T$ and therefore the premises are true. But since $V(C) = T$, then $V(\neg C) = F$.

Problem 3. Let α be the statement:

$$\forall x \in \mathbb{Z}.\Big(\forall y \in \mathbb{Z}.\big((x < y) \Rightarrow (\exists z \in \mathbb{Z}.x < z + 1 < y)\big)\Big)$$

a. present $\neg\alpha$ without the "$\neg$" symbol.

   Solution.

$$\forall x \in \mathbb{Z}.\Big(\forall y \in \mathbb{Z}.\big((x < y) \Rightarrow (\exists z \in \mathbb{Z}.x < z + 1 < y)\big)\Big) \equiv$$

$$\exists x \in \mathbb{Z}\Big(\exists y \in \mathbb{Z}\big((x < y) \wedge (\forall z \in \mathbb{Z}(x \geq z + 1 \vee z + 1 \geq y)\big)\Big)$$

b. Prove or disprove α.

   Disprove: Define $x = 0$ and $y = 1$. The $x < y$ and let $z \in \mathbb{Z}$, either $z + 1 \in \mathbb{Z}$ so either $z + 1 \geq 1$ or $z \leq 0$.

Problem 4. Prove that if $2a - b$ is multiple of n and $a - b$ is multiple of n, then a is multiple of n.

solution. Suppose that $2a - b$ and $a - b$ are multiples of n, then there are c, d integers such that $2a - b = cn$ and $a - b = dn$. Hence

$$a = 2n - b - (a - b) = cn - dn = n(c - d)$$

Since $c - d$ is an integer, a is divisible by $c - d$.

Problem 5. Prove that if $n^2 + 2n + 11$ is odd, then n is even.

solution Let us prove the contrapositive, suppose that n is odd. WTP, $n^2 + 2n + 11$ is even. By assumption, $n = 2k + 1$. Substituting for n, we get

$$n^2+2n+11 = (2k+1)^2+2(2k+1)+11 = 4k^2+4k+1+4k+2+11 = 2(2k^2+4k+7)$$

Since $2k^2 + 4k + 7$ is an integer, $n^2 + 2n + 11$ is even.

Problem 6. Express the following sets using the list principle. No proof required.

1. $\{x \in \mathbb{N} \mid x \cdot 5 \leq 5\}$

   solution $\{x \in \mathbb{N} \mid x \cdot 5 \leq 5\} = \{0, 1\}$

2. $\{x \in \mathbb{Z} \mid |x| < 10 \land \exists y.5y = x\}$.

   Solution $\{x \in \mathbb{Z} \mid |x| < 10 \land \exists y.5y = x\} = \{-5, 0, 5\}$.

3. $\Big\{\{x^2, |x|\} \mid x \in \{-1, 0, 1, 2\}\Big\}$

   Solution. $\Big\{\{x^2, |x|\} \mid x \in \{-1, 0, 1, 2\}\Big\} = \{\{1\}, \{0\}, \{2\}, \{4, 2\}\}$

Problem 7. Prove the following statements:

1. $\frac{1}{2} \in \{q + 1 \mid q \in \mathbb{Q}\}$.

   proof. By the replacement principle WTP $\exists q \in \mathbb{Q}\ q + 1 = \frac{1}{2}$. Define $q = -\frac{1}{2}$, then $q \in \mathbb{Q}$ and $q + 1 = \frac{1}{2}$. Hence $\frac{1}{2} \in \{q + 1 \mid q \in \mathbb{Q}\}$.

2. $3 \notin \{x \in \mathbb{Z} \mid x^2 + 2x + 1 = 0\}$

   proof. We have $3^2 + 6 + 1 \neq 0$ and by the separation principle $3 \notin \{x \in \mathbb{Z} \mid x^2 + 2x + 1 = 0\}$.

3. $5 \in \Big\{n \in \mathbb{N} \mid n^2 \in \{2m + 1 | m \in \mathbb{N}\}\Big\}$

   proof. By the separation principle WTP $5 \in \mathbb{N} \land 5^2 \in \{2m+1 \mid m \in \mathbb{N}\}$. Clearly $5 \in \mathbb{N}$, and for $m = 12$ we have that $5^2 = 25 = 2 \cdot 12 + 1$ thus by the replacement principle $5^2 \in \{2m + 1 \mid m \in \mathbb{N}\}$. Hence $5 \in \Big\{n \in \mathbb{N} \mid n^2 \in \{2m + 1 | m \in \mathbb{N}\}\Big\}$