Homework 8

MATH 300 (due April 5) March 29, 2024

Problem 1. Prove that if $f : A \to B$, $g : B \to C$ are surjections then $g \circ f$ is a surjection.

Solution. Suppose that f, g are surjective WTP $g \circ f$ is subjective. Let $c \in C$, since g is subjective there is $b \in B$ such that $g(b) = c$ and since f is subjective there is $a \in A$ such that $f(a) = b$. Hence $c = g(b) = g(f(a)) = g \circ f(a)$.

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Problem 2. Prove or disprove the following items:

1. If $f : A \longrightarrow B$ is injective, then for every $X \subseteq A$, $f \restriction X$ is injective.

2. If $f : A \longrightarrow B$ is surjective, then for every $X \subseteq A$, $f \restriction X$ is surjective.

Solution.

1. The statement is true. Proof: Let $f : A \longrightarrow B$ be an injective function, and $X \subseteq A$. We want to prove that $f \restriction X$ is injective. So, let $x_1, x_2 \in X$ such that $(f \restriction X)(x_1) = (f \restriction X)(x_2)$ WTP $x_1 = x_2$. As $\forall x \in X, (f \restriction X)(x) = f(x)$, it follows that $f(x_1) = f(x_2)$. By our assumption, f is injective, so $f(x_1) = f(x_2)$, implies that $x_1 = x_2$. Therefore, $(f \restriction X)$ is injective.

2. The statement is false. Proof: Let $A = \{1, 2\}$ $B = \{1, 2\}$ $f = id_{\{1,2\}}$ now let $X = \{1\}$, then $f \restriction \{1\}$ is not onto B since 2 is not the image of 1.

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Problem 3. Prove that if $f : A \longrightarrow B$ is a function such that for some $X \subsetneq A$, $f \upharpoonright X : X \longrightarrow B$ is onto B, then f is not injective.

Solution. Let $f : A \longrightarrow B$ be a function and $X \subsetneq A$ such that $f \upharpoonright X : X \longrightarrow B$ is surjective. We want to prove that f is not injective. Towards a contradiction, suppose f is injective. Because X is a proper subset of A, there exists some element $a_0 \in A$ such that $a_0 \notin X$. Let $b_0 = f(a_0)$. As $f \upharpoonright X$ is surjective, then for all $b \in B$, there exists some $x \in X$ such that $x = (f \upharpoonright X)(b)$. So let $x_0 \in X$ such that $b_0 = (f \upharpoonright X)(x_0)$. Then $b_0 = f(x_0) = f(a_0)$. Because f is injective, $x_0 = a_0$, and thus $a_0 \in X$, which is a contradiction. Therefore, f is not injective.

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Problem 4. For each of the following functions, determine if it is injective/ surjective and prove your answer.

1. $f_1 : \mathbb{R} \to \mathbb{R}$, defined by $f_1(x) = 5x - x^2$.
2. $f_2 : \mathbb{R} \to P(\mathbb{R})$, defined by $f_2(x) = \{x^2\}$.
3. $f_3 : P(\mathbb{R}) \to P(\mathbb{N})$, defined by $f_3(x) = x \cap \mathbb{N}$.
4. $f_4 : P(\mathbb{N}) \to \mathbb{N}$, defined by $f_4(x) = \begin{cases} \min(x) & 4 \in x \\ 0 & else \end{cases}$.
5. $f_5 : P(\mathbb{R}) \to P(\mathbb{N}) \times P(\mathbb{Z}) \times P(\mathbb{Q})$, defined by
$$f_5(X) = \langle X \cap \mathbb{N}, X \cap \mathbb{Z}, X \cap \mathbb{Q} \rangle$$
6. $f_6 : \mathbb{N} \times \mathbb{Z} \to P(\mathbb{N})$, defined by $f_6(\langle n, m \rangle) = \{x \in \mathbb{N} \mid n < x < m\}$.

Solution

1. f_1 is not injective nor surjective. Proof:
   (a) $f_1(0) = 0 = f_1(5)$. Clearly $0 \neq 5$, so f_1 is not injective.
   (b) There exists $y \in \mathbb{R}$ such that $\forall x \in \mathbb{R}, f(x) \neq y$. In particular, let $y = 8$. The equation $8 = 5x - x^2$ has no real solution. So, $\forall x \in \mathbb{R}, f(x) \neq 8$. Therefore, f_1 is not surjective.

2. f_2 is not injective or surjective. Proof:
   (a) $f_2(1) = \{1\} = f_2(-1)$. Clearly $1 \neq -1$, so f_2 is not injective.
   (b) Consider the set $\{1, 2\} \subseteq P(\mathbb{R}$. Note that for all x, $|f_2(x)| = 1$, but $|\{1, 2\}| = 2$. So $\forall x \in \mathbb{R}, f x^2 \neq \{1, 2\}$, and thus f_2 is not surjective.

3. f_3 is surjective but not injective. Proof:

   (a) $f_3(\{1.5\}) = \emptyset = f_3(\{1.1\})$, but $\{1.5\} \neq \{1.1\}$. Therefore, f_3 is not injective.

   (b) Let $Y \in P(\mathbb{N})$, and $X = Y$. Then $X \subseteq P(\mathbb{R})$, and $f_3(X) = X \cap \mathbb{N} = X$. Therefore, f_3 is surjective.

4. f_4 is not injective or surjective. Proof:

   (a) $f_4(\{1\}) = 0 = f_4(\{2\})$, but $\{1\} \neq \{2\}$. Therefore, f_4 is not injective.

   (b) Let y be a natural number greater than 4, and let $X \subseteq \mathbb{N}$. Cases:

      i. $4 \in X$. Then $\min(X) \leq 4$, and so $f_4(X) < y$.

      ii. $4 \notin X$. Then $f_4(X) = 0 \neq y$.

      Therefore, f_4 is not surjective.

5. f_5 is not injective or surjective. Proof:

   (a) $f_5(\{\pi\}) =< \emptyset, \emptyset, \emptyset >= f_5(\{\sqrt{2}\})$, but $\{\pi\} \neq \{\sqrt{2}\}$. Therefore, f_5 is not injective.

   (b) Let $Y =< \{1\}, \{-1\}, \{\frac{1}{2}\} >$. Towards a contradiction, suppose f_5 is surjective. Then there exists some $X \in P(\mathbb{R})$ such that $f_5(X) = Y$. By the definition of f, for some $N \subseteq \mathbb{N}$, $X \cap \mathbb{N} = \{1\}$. Thus, $1 \in X$. However, for some $Z \subseteq \mathbb{Z}$, $X \cap \mathbb{Z} = \{-1\}$. Thus, $1 \notin \mathbb{Z}$, which is a contradiction. Therefore, for all $X \in P(\mathbb{R})$, $f_5(X) \neq Y$, so f_5 is not surjective.

6. f_6 is not injective or surjective. Proof:

(a) $f_6(< 1, -1 >) = \emptyset = f_6(< 1, -2 >)$, but $< 1, -1 > \neq < 1, -2 >$. Therefore, f_6 is not injective.

(b) Let $Y = \{0\}$ and $X \in \mathbb{N} \times \mathbb{Z}$. Towards a contradiction, suppose $f_6(X) = \{0\}$. Then by the separation principle, $n < 0 < m$, where $n \in \mathbb{N}, m \in \mathbb{Z}$. Then n is a natural number < 0, which is a contradiction. Thus, $\forall \mathrm{X} \in \mathbb{N} \times \mathbb{Z}$, $f_6(X) \neq Y$. Therefore, f_6 is not surjective.

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Problem 5. In the following items, no proof required (just a formal definition of the functions):

1. Find an injective function $f : \mathbb{N} \to P(\mathbb{N})$.

   Solution. $f(n) = \{n\}$

2. Find a surjective function $f : \mathbb{Z}^2 \to \mathbb{Q}$.

   Solution. $f(\langle z_1, z_2 \rangle) = \begin{cases} 0 & z_2 = 0 \\ \frac{z_1}{z_2} & o.w \end{cases}.$

3. (*Optional) Find an injective function $f : \mathbb{R} \to P(\mathbb{Q})$ [Hint: Use the density of the rationals inside the reals].

   Solution. $f(r) = \{q \in \mathbb{Q} \mid q < r\}$.

4. Find a surjective function $f : \mathbb{N} \to \mathbb{Z}$.

   Solution. $f(n) = (-1)^n \lfloor \frac{n}{2} \rfloor$