Homework 5-Sols

MATH 300 (due Oct 18) Oct 11, 2024

Problem 1. Prove that for any two sets A, B, $A = B$ if and only if $A \Delta B = \emptyset$

Solution: Let us prove this by a double implication:

$\Rightarrow$ Suppose that $A = B$, then $A \Delta B = A \Delta A = \emptyset$ by the set identities table.

$\Leftarrow$ Let us move to the contrapositive. Suppose that $A \neq B$, and let us prove that $A \Delta B \neq$. Let us split into cases:

(1) If there is $x \in A$ such that $x \notin B$, then $x \in A \setminus B$ and by definition of symmetric difference $x \in A \Delta B$. Therefore $A \Delta B \neq \emptyset$

(2) The case that there is $x \in B$ and $x \notin A$ is symmetric.

In any case $A \Delta B \neq \emptyset$.

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Problem 2. Compute the following sets. No proof required.

1. $\Big\{a + b : a \in \{0,5\}, b \in \{2,4\}\Big\} \setminus \{7,10\} = \{2,4,9\}$.

2. $(1,3) \cup [2,4) = (1,4)$

3. $\mathbb{Z} \cap [0,\infty) = \mathbb{N}$

4. $\mathbb{N}_{even} \Delta \mathbb{N}_{+} = \{0\} \cup \mathbb{N}_{odd}$

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Problem 3. Let X and Y be sets.

(i) Prove that $Y \setminus (Y \setminus X) = X \cap Y$.

(ii) Prove that $X \subseteq Y$ if and only if $X \cup Y = Y$.

Solution

(i) Let us prove this by a double inclusion:

$\subseteq$ Let $x \in Y(Y \setminus X)$. WTP $x \in X \cap Y$. By assumption, $x \in Y$ and $x \notin Y \setminus X$a and since $x \in Y$, it must follow that $x \in X$ and therefore $x \in X \cap Y$.

$\supseteq$ Let $x \in X \cap Y$. WTP $x \in Y(Y \setminus X)$. Indeed, $x \in X$ and $x \in Y$ and therefore $x \notin YX$. By definition of difference $x \in Y \setminus (Y \setminus X)$.

Since we proved a double inclusion we conclude that $Y \setminus (Y \setminus X) = X \cap Y$.

(ii) Let us prove a double implication.

$\Rightarrow$ Suppose that $X \subseteq Y$. WTP $X \cup Y = Y$, We will prove this set equality by a double inclusion. Inclusion from right to left is clear. For the other direction, let $x \in X \cup Y$, if $x \in Y$, then we are done. If $x \in X$, then since $X \subseteq Y$ then $x \in Y$ and again we are done.

$\Leftarrow$ suppose that $X \cup Y = Y$ and let us prove that $X \subseteq Y$. Let $x \in X$, WTP $x \in Y$. It follows that $x \in X \cup Y$, and since $X \cup Y = Y$, then $x \in Y$

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Problem 4. Prove that if $A \cap B \subseteq C$ and $x \in A \setminus C$, then $x \notin B$.

[Hint: Prove it by contradiction.]

Solution Suppose that $A \cap B \subseteq C$ and $x \in A \setminus C$. WTP $x \notin B$. Suppose towards a contradiction that $x \in B$. By our assumption $x \in A$ and $x \notin C$, therefore $x \in A \cap B$. Since $A \cap B \subseteq C$, then $x \in C$, this is a contradiction.