Homework 3-Sols

MATH 300 (due Feb 16) Feb 9, 2022

Problem 1. Apply each of the following claims to two specific examples of your choice or find a counterexample. In your solution, you should provide the examples and what you have concluded from the statements:

a. Suppose that n is a integer, then 24 divides $n(n+1)(n+2)(n+3)$.

b. Suppose that x, y, z are three integers such that $x^2 + y^2 = z^2$, then either 3 divides x or 3 divides y.

Solution. [a.] for example for $n = 1,$, $1 \cdot 2 \cdot 3 = 6$ is divisible by 6 and therefore $1 \cdot 2 \cdot 3 \cdot 4$ is divisible by 24. also $2 \cdot 3 \cdot 4 = 24$ is divisible by 6 and therefore $2 \cdot 3 \cdot 4 \cdot 5$ is divisible by 24,

[b.] Counter example $5^2 + 12^2 = 13^2$ but $5, 12$ are not divisible by 3.

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Problem 2. Prove the following equivalences (using a double implication):

An integer is divisible by 4 if and only if its last two digits form a number divisible by 4.

[Hint: Decompose $n = 100l + d$ where k, l is some integers and $0 \leq d \leq 99$. Then the number d is the last two digits.]

Solution Let $n = 100k + d$ where $0 \leq d \leq 99$. Let us prove this equivalence using a double implication

- $\rightarrow$ Suppose that n is divisible by 4. WTP d is divisible by 4. indeed, $d = n - 100k$, and since both n and $100k$ are divisible by 4, n is divisible by 4, by a theorem we saw in class that the difference of two numbers divisible by m is divisible by m.
- $\leftarrow$ Suppose that d is divisible by 4 then $n = 100k + d$ is a sum of two numbers divisible by 4, hence divisible by 4.

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Problem 3. Prove that if a and b are odd integers, then $a^2 - b^2$ is a multiple of 8.

Solution Suppose that a, b are odd. WTP that $a^2 - b^2$ is divisible by 8. By assumption, there are k, l integers such that $a = 2k + 1$ and $b = 2l + 1$. It follows that

$$a^2 - b^2 = (4k^2 + 4k + 1) - (4l^2 + 4l + 1) = 4(k^2 + k - (l^2 + l))$$

In class we proved that for every n, n^2+n is even and therefore $k^2+k-(l^2+l)$ is the difference if two even number hence even. Hence there is an integer r such that $k^2 + k - (l^2 + l) = 2r$ and therefore

$$a^2 - b^2 = 4(k^2 + k - (l^2 + l)) = 8r$$

Therefore $a^2 - b^2$ is divisible by 8.

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Problem 4. Let a, b, c be integers. Prove that if $a^2 + b^2 = c^2$, then abc is even.

Solution. Let us prove the contrapositive. WTP $a^2 + b^2 = c^2$ Suppose that abc is odd. Then a, b, c must all be odd. But then a^2, b^2, c^2. Now $a^2 + b^2$ is even as the sum of two odds. We conclude that $a^2 + b^2 \neq c^2$, since a number cannot be both even and odd.

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Problem 5. (1) Prove that for every rational number $q \in \mathbb{Q}$ $q \neq 0$, $\sqrt{2} \cdot q$ is irrational.

Soltuion. Suppose towards contradiction that $\sqrt{2}q = p \in \mathbb{Q}$. Then $\sqrt{2} = \frac{p}{q}$. The ration of two rationals is rationals and therefore $\sqrt{2} \in \mathbb{Q}$, contradicting the theorem we saw in class.

(2) Prove or disprove: the sum or irrational numbers is irrational.

solution. Counterexample, $\sqrt{2} + (1 - \sqrt{2}) = 1$.

(3) Prove that $\sqrt{5}$ is irrational.

Soltuion. Like in WS4.

(4) (optional) Formulate a conjecture for the rationality and irrationality of real numbers of the form $\sqrt{n}$.

Solution For any natural number n, either $\sqrt{n}$ is an integer or irrational.