Homework 10

MATH 300 | (due April 19) | April 12, 2024

Problem 1. Show that $P(\mathbb{N}) \times P(\mathbb{N}) \approx P(\mathbb{N})$.

[Hint: Use the interleaving function exercise from the previous HW.]

Solution. $P(\mathbb{N}) \times P(\mathbb{N}) \sim^{(*)+(***)} {}^{\mathbb{N}}\{0,1\} \times {}^{\mathbb{N}}\{0,1\} \sim^{(**)} {}^{\mathbb{N}}\{0,1\} \sim^{(***)} P(\mathbb{N})$.

$(*)$– we saw in class that $A \sim A'$ and $B \sim B'$ then $A \times B \sim A' \times B'$.

$(**)$– the previous homework.

$(***)$– we saw in class that ${}^{\mathbb{N}}\{0,1\} \sim P(\mathbb{N})$.

Problem 2. Prove that $P(\mathbb{Z}\times\mathbb{Z})\times P(\mathbb{Z}) \sim P(\mathbb{N})$.

Solution. $P(\mathbb{Z}\times\mathbb{Z})\times P(\mathbb{Z}) \sim^{(*)} P(\mathbb{N})\times P(\mathbb{Z}) \sim^{(**)} P(\mathbb{N})\times P(\mathbb{N}) \sim^{(***)} P(\mathbb{N})$.

$(*)-$ $\mathbb{Z}\times\mathbb{Z} \sim \mathbb{N}\times\mathbb{N} \sim \mathbb{N}$ and if $A \sim A'$ then $P(A) \sim P(A')$.

$(**)-$ $\mathbb{N} \sim \mathbb{Z}$ and if $A \sim A'$ then $P(A) \sim P(A')$.

$(***)-$ the previous exersice.

Problem 3. Prove that if $A \sim A'$ and $B \sim B'$ are sets such that $A \cap B = A' \cap B' = \emptyset$ then $A \cup B \sim A' \cup B'$.

Solution. Let $f : A \to A'$ be a bijection and $g : B \to B'$ be a bijection. Define $h : A \cup B \to A' \cup B'$ by

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

Note that h is well defined since $A \cap B = \emptyset$. To see that h is one-to-one let $x, y \in A \cup B$ be such that $h(x) = h(y)$. Let us split into cases:

(1) if $h(x) \in A'$, then since $A' \cap B' = \emptyset$, we have that $x, y \in A$ and therefore $f(x) = h(x) = h(y) = f(y)$, and since f is one-to-one, $x = y$.

(2) if $h(x) \in B'$, this is similar using he fact that g is one-to-one.

To see that h is onto, let $c \in A' \cup B'$. If $c \in A'$, since f is onto, there is $a \in A$ such that $h(a) = f(a) = c$. Similarly, if $c \in B'$ there is $a \in B$ such that $h(a) = g(a) = c$. in any case there is $x \in A \cup B$ such that $h(x) = c$ and therefore h is onto.

Problem 4. Prove that for any function $f : A \to B$, $|f| = |A|$. [Remark: recall that a function is a set of ordered pairs.]

Solution. Define $F : A \to f$ by $F(a) = \langle a, f(a) \rangle$. Let us show that F is a bijection. Let a, a' be such that $F(a) = F(a')$, then $\langle a, f(a) \rangle \langle a', f(a') \rangle$ and in particular $a = a'$. to see that F is onto, any element in f is of the form $\langle a, f(a) \rangle$ for some a and therefore $F(a) = \langle a, f(a) \rangle$.