MATH 300: CHAPTER 5- EQUINOUMERABILITY

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Definition 0.1. Let A, B be any sets. We say that:

(1) $A \sim B$ "A and B are equinumerable" if there is a bijection $f : A \to B$.
(2) $A \preceq B$ "A is at most the size of B" if there is an injective function $f : A \to B$.
(3) $A \not\sim B$ if $\neg(A \sim B)$, namely if there is no bijection $f : A \to B$.
(4) $A \prec B$ if $A \preceq B$ and $A \not\sim B$.

Example 0.2. (1) $\{1, 2, 3\} \sim \{2, 7, 19\}$ as witnessed by the bijection

$$f(x) = \begin{cases} 2 & x = 1 \\ 7 & x = 2 \\ 19 & x = 3 \end{cases}$$

(2) $\mathbb{N} \sim \mathbb{N}_{even}$ as witnessed by the function $f : \mathbb{N} \to \mathbb{N}_{even}$, $f(n) = 2n$.
(3) $A \preceq P(A)$ for every set A as witnessed by the function $f : A \to P(A)$, $f(a) = \{a\}$.
(4) $(0, 1) \sim (1, 3)$ as given by $f : (0, 1) \to (1, 3)$, $f(x) = 2x + 1$.
(5) $\{X \in P(\mathbb{N}) \mid 0 \in X\} \sim P(\mathbb{N})$ by $f : P(\mathbb{N}) \to \{X \in P(\mathbb{N}) \mid 0 \in X\}$, $f(X) = \{0\} \cup \{x + 1 \mid x \in X\}$.
(6) $\mathbb{N} \times \mathbb{N} \preceq P(\mathbb{N})$ witnessed by $f : \mathbb{N} \times \mathbb{N} \to P(\mathbb{N})$, $f(\langle n, m\rangle) = \{n, n + m\}$.
(7) $A \subseteq B \to A \preceq B$ as witnessed by the function $f : A \to B$, $f(a) = a$.
(8) Clearly $A \sim B$ implies $A \preceq B$.

Claim 0.2.1. *for any sets A, B, C:*

(1) $A \sim A$.
(2) $A \sim B \to B \sim A$.
(3) $A \sim B \wedge B \sim C \to A \sim C$ *and* $A \preceq B \preceq C \to A \preceq C$.

Are there two infinite sets which are not equinumerable?

Proposition 0.3. $\mathbb{N} \sim \mathbb{Z}$

Proof. Define $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} \frac{n}{2} & n \in \mathbb{N}_{even} \\ -\frac{n+1}{2} & n \in \mathbb{N}_{odd} \end{cases}$$

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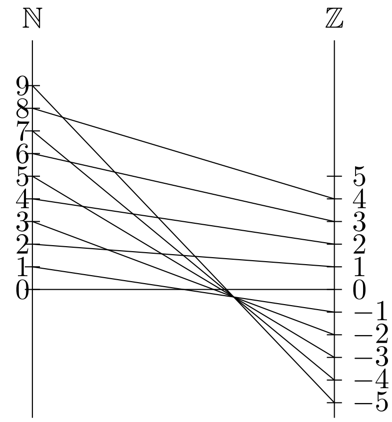


□

$\mathbb{Z}$ is like "two copies" of $\mathbb{N}$. What about infinitely many copies of $\mathbb{N}$? $\mathbb{N}\times\mathbb{N}$.

Proposition 0.4. $\mathbb{N}\sim\mathbb{N}\times\mathbb{N}$

Proof. Define $f:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ by $f(\langle n,m\rangle)=2^n(2m+1)-1$. □

We will have an easier proof later.

Proposition 0.5. *Let A,A',B,B' be sets such that $A\sim A'$ and $B\sim B'$. Then:*

(1) $P(A)\sim P(A')$.
(2) $A\times B\sim A'\times B'$.
(3) ${}^BA\sim {}^{B'}A'$.
(4) If A,B are disjoint and A',B' are disjoint then $A\uplus B\sim A'\uplus B'$.

The above proposition is true upon replacing $\sim$ by $\preceq$ everywhere.

Proof. Let us prove for example (1). Let $f:A\to A'$ be a bijection. One should check that $F:P(A)\to P(A')$ defined by $F(X)=f''X$ is a bijection. □

Example 0.6. $\mathbb{N}\sim\mathbb{Z}\times\mathbb{Z}$.

What about $\mathbb{Q}$? clearly $\mathbb{N}\preceq\mathbb{Q}$

Claim 0.6.1. (AC) *Suppose that $A\neq\emptyset$. Then $A\preceq B$ iff there is $f:B\to A$ onto.*

Proof. Suppose that $g:A\to B$ is one-to-one. Let us $a^*\in A$ be some elements. Define $f:B\to A$ by

$$f(b)=\begin{cases}a^* & b\notin Im(g)\\ g^{-1}(b) & b\in Im(g)\end{cases}$$

This is well defined since g is invertible on its image. For the other direction, suppose that $f:B\to A$ is onto. Let us define $g:A\to B$ one-to-one. For every $a \in A$, since f is onto, there is some (choose!) $b_a \in f^{-1}[\{a\}]$. Define $g(a) = b_a$. Then g is one to one since if $a \neq a'$ then $b_a \in f^{-1}[\{a\}]$ and $b_{a'} \in f^{-1}[\{a'\}]$ which are disjoint sets and therefore $b_a \neq b_{a'}$. Hence g is one-to-one. □

Example 0.7. $\mathbb{Q} \preceq \mathbb{Z} \times \mathbb{Z} \sim \mathbb{N}$. The function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ defined by

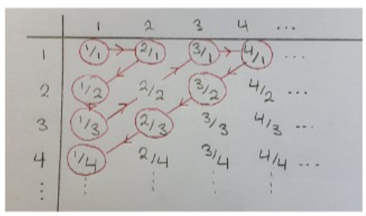
$$f(\langle z_1, z_2\rangle) = \begin{cases} \frac{z_1}{z_2} & z_2 \neq 0 \\ 0 & else \end{cases}$$

is onto

So we are in the situation where $\mathbb{N} \preceq \mathbb{Q}$ and $\mathbb{Q} \preceq \mathbb{N}$. Does it mean that $\mathbb{N} \sim \mathbb{Q}$? Yes! but this requires a highly non-trivial theorem which we will prove later. Instead, let us give direct proof:

Theorem 0.8. $\mathbb{N} \sim \mathbb{Q}$

"Proof". We are about to construct a function $f : \mathbb{N}_+ \to \mathbb{Q}_+ = \{q \in \mathbb{Q} \mid q > 0\}$ one-to-one and onto, by recursion on $\mathbb{N}_+$. To do so, we think of the $\mathbb{Q}_+$ as elements in the matrix $\mathbb{N}_+ \times \mathbb{N}_+$



We go by induction on the diagonal rows (namely pair $\langle k_1, k_2\rangle$ such that $k_1 + k_2 = n$ starting at $n - 2$). We define $f(1) = 1/1 = 1$. Suppose we reached the n^{th} row. In row $n+1$, we keep defining f on new (finitely many) values only for those pairs which represent a rational number which haven't appeared before (to ensure the function is one-to-one). The resulting function f is a bijection from $\mathbb{N}_+$ to $\mathbb{Q}_+$. Let us now define a function $g : \mathbb{N} \to \mathbb{Q}$ by

$$g(n) = \begin{cases} 0 & n = 0 \\ f(\frac{n}{2}) & n \in \mathbb{N}_{even} \setminus \{0\} \\ -f(\frac{n+1}{2}) & n \in \mathbb{N}_{odd} \end{cases}$$

□

So far we failed to find two infinite sets which are not equinumerable.

Theorem 0.9. *(AC) If A is infinite then $\mathbb{N} \preceq A$.*

Proof. We construct the function $f : \mathbb{N} \to A$ by recursion, there is always a possibility to continue the definition of f and pick a new element since otherwise, A was finite. □

Definition 0.10. A set A is countable if $A \sim \mathbb{N}$. A is uncountable if $\mathbb{N} \prec A$.

Theorem 0.11. *The following sets are countable:* $\mathbb{Z}, \mathbb{N}_{even}, \mathbb{Q}, \mathbb{N}\times\mathbb{N}, \mathbb{N}^n (n \geq 1)$

Proof. It remains to show that $\mathbb{N}^n$ is countable. We prove that by induction on n. For $n = 1$, this is clear. Suppose that $\mathbb{N}^n \sim \mathbb{N}$, then

$$\mathbb{N}^{n+1} \sim \mathbb{N}^n \times \mathbb{N} \sim \mathbb{N} \times \mathbb{N} \sim \mathbb{N}.$$

□

Theorem 0.12 (Cantor's Diagonalization Theorem)**.** $\mathbb{N} \prec {}^{\mathbb{N}}\{0,1\}$

Proof. It is not hard to prove that $\mathbb{N} \preceq {}^{\mathbb{N}}\{0,1\}$. So it remains to prove that $\mathbb{N} \not\sim {}^{\mathbb{N}}\{0,1\}$. Assume toward a contradiction that $F : \mathbb{N} \to {}^{\mathbb{N}}\{0,1\}$ was onto. Let us show how to produce a function $g : \mathbb{N} \to \{0,1\}$ (i.e. an element in the range of F) such that for every n, $F(n) \neq g$ (i.e. g is not in the image of F). This will produce a contradiction to the assumption that F is onto.

For each n, $F(n) : \mathbb{N} \to \{0,1\}$ so we write it as a binari sequence

$$f_n := F(n) = \langle F(n)(0), F(n)(1), F(n)(2), ...\rangle$$

So the list of functions $F(0), F(1), F(2)$ can be written in a matrix:

$$\begin{array}{ccccccc}
\underline{f_0(0)} & f_0(1) & f_0(2) & f_0(3) & \dots & f_0(n) & \dots \\
f_1(0) & \underline{f_1(1)} & f_1(2) & f_1(3) & \dots & f_1(n) & \dots \\
f_2(0) & f_2(1) & \underline{f_2(2)} & f_2(3) & \dots & f_2(n) & \dots \\
f_3(0) & f_3(1) & f_3(2) & \underline{f_3(3)} & \dots & f_3(n) & \dots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \dots & \ddots \\
f_n(0) & f_n(1) & f_n(2) & f_n(3) & \dots & \underline{f_n(n)} & \dots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots
\end{array}$$

Note that each value in this matrix is 0 or 1. We would like to define a function $g : \mathbb{N} \to \{0,1\}$, namely a binary sequence $\langle g(0), g(1), g(2), ...\rangle$ such that g defers from each row at some n. so we change the values from 0 to 1, Start by setting $g(0) = 0$ if $f_0(0) = 1$ or $g(0) = 1$ if $f_0(0) = 0$ ("flip the bit") algebraically we can write that as $1 - f_0(0)$. Moving to f_1, we flip the value $f_1(1)$ and define $g(1) = 1 - f_1(1)$. In general, we flip the values on the diagonal and define $g(n) = 1 - f_n(n)$. To that g is as wanted, suppose toward a contradiction that $g = f_n$ for some n, then by function equality we get that $1 - f_n(n) = g(n) = f_n(n)$ hence $f_n(n) = \frac{1}{2}$, contradiction. □

Corollary 0.13. *For every set* A*,* $A \prec {}^{A}\{0,1\}$*.*

Proof. If $A = \emptyset$ this is straightforward. So assume $A \neq \emptyset$. Toward a contradiction, suppose that $F : A \to {}^{A}\{0,1\}$ is onto and denote by $f_a = F(a)$. Define $g : A \to \{0,1\}$ by

$$g(a) = 1 - f_a(a)$$

The continuation is as before. □

Theorem 0.14. $P(A) \sim {}^{A}\{0,1\}$

Proof. For a subset $B \subseteq A$ we define the indicator function $\chi^A_B : A \to \{0,1\}$ by

$$\chi^A_B(a) = \begin{cases} 1 & a \in B \\ 0 & a \notin B \end{cases}$$

The function $\chi^A : P(A) \to {}^A\{0,1\}$ defined by $\chi^A(B) = \chi^A_B$ is a bijection (prove that!). □

Theorem 0.15 (Cantor's Theorem)**.** $A \prec P(A)$

Proof. $a \mapsto \{a\}$ is an injection from A to $P(A)$ hence $A \preceq P(A)$. Suppose toward a contradiction that $A \sim P(A)$, then by the previous theorem $A \sim {}^A\{0,1\}$, contradiction. □

Corollary 0.16. $\mathbb{N} \prec P(\mathbb{N}) \prec P(P(\mathbb{N})) \prec ...$

Theorem 0.17 (Cantor-Schröeder-Bernstein-No proof)**.** *Let A, B be sets and supose that $A \preceq B \wedge B \preceq A$ then $A \sim B$.*

Example 0.18. Prove that ${}^{\mathbb{N}}\mathbb{N} \sim P(\mathbb{N})$

Proof. On one hand we have $P(\mathbb{N}) \sim {}^{\mathbb{N}}\{0,1\} \preceq {}^{\mathbb{N}}\mathbb{N}$ (the last equality is due to inclusion) on the other hand we have ${}^{\mathbb{N}}\mathbb{N} \subseteq P(\mathbb{N} \times \mathbb{N}) \sim P(\mathbb{N})$. So by Cantor-Schroeder-Berstein $P(\mathbb{N}) \sim {}^{\mathbb{N}}\mathbb{N}$. □

Theorem 0.19. $\mathbb{R} \sim {}^{\mathbb{N}}\{0,1\}$

"“Proof”. On one hand we have that every $x \in \mathbb{R}$ is a Dedekind cut so $x \in P(\mathbb{Q})$ and therefore

$$\mathbb{R} \preceq P(\mathbb{Q}) \sim P(\mathbb{N}) \sim {}^{\mathbb{N}}\{0,1\}$$

For the other direction, we will define a function $F : {}^{\mathbb{N}}\{1,2\} \to \mathbb{R}$ defined by

$$F(f) = 0.f(0)f(1)f(2)...$$

is one-to-one as every decimal representation is not eventually 0. Also it is clear that $\{0,1\} \sim \{1,2\}$ hence

$${}^{\mathbb{N}}\{0,1\} = {}^{\mathbb{N}}\{1,2\} \preceq \mathbb{R}$$

By Cantor- Schroeder-Berstein, $\mathbb{R} \sim {}^{\mathbb{N}}\{0,1\}$ □

In particular $\mathbb{R}$ is uncountable.

Problem 1. *Prove that ${}^{\mathbb{N}}\{0,1\} \times {}^{\mathbb{N}}\{0,1\} \sim {}^{\mathbb{N}}\{0,1\}$ [Hint: consider the interweaving function that take two binary sequences $\langle a_0, a_1, ...\rangle, \langle b_0, b_1, ...\rangle$ and outputs $\langle a_0, b_0, a_1, b_1, a_2, b_2, ...\rangle$]*

About this result, Cantor said: “My eyes can see it but I cannot believe it”.

Theorem 0.20. *for every $n \geq 1$, $\mathbb{R}^n \sim \mathbb{R}$.*

Proof. It suffices to prove that $\mathbb{R} \times \mathbb{R} \sim \mathbb{R}$ and then the same inductive argument as with the case of the natural numbers will work. Indeed,

$$\mathbb{R} \times \mathbb{R} \sim {}^{\mathbb{N}}\{0,1\} \times {}^{\mathbb{N}}\{0,1\} \sim {}^{\mathbb{N}}\{0,1\} \sim \mathbb{R}$$

□

Theorem 0.21. *For every $\alpha < \beta$ reals $[\alpha, \beta] \sim (\alpha, \beta) \sim (\alpha, \infty) \sim \mathbb{R}$*

Proof. First we note that $tn : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is one-to-one and onto hence $(-\frac{\pi}{2}, \frac{\pi}{2}) \sim \mathbb{R}$. Since $(-\frac{\pi}{2}, \frac{\pi}{2}) \subseteq [-\frac{\pi}{2}, \frac{\pi}{2}] \subseteq (\frac{\pi}{2}, \infty) \subseteq \mathbb{R}$ we also have that all those sets are equinumerable. Now it is not hard to find bijections of the from $f(x) = ax + b$ which moves (α, β) to $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $[\alpha, \beta]$ to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and (α, ∞) to $(-\frac{\pi}{2}, \infty)$. □

Definition 0.22. The continuum hypothesis (CH): Every set $A \subseteq \mathbb{R}$ is either finite, countable, or is equinumerable to the reals.

Theorem 0.23 (Godel and Cohen)**.** *The continuum hypothesis cannot be proven nor refuted from ZFC.*

Theorem 0.24. *The countable union of at most countable sets is at most countable*

Proof. Let A_n be a sequence of sets such that for each n, A_n is at most countable. Let us define B_n as follows, $B_0 = A_0$ and $B_{n+1} = A_{n+1} \setminus (\cup_{k=0}^{n} A_k)$. Since $B_n \subseteq A_n$, our assumption that A_n is at most countable implies that there is $f_n : B_n \to \mathbb{N}$ which is one-to-one. Note that if $n \neq m$ then $B_n \cap B_m = \emptyset$ and also that $\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} B_n$. Define $g : \bigcup_{n \in \mathbb{N}} A_n \to \mathbb{N} \times \mathbb{N}$ by $g(n) = \langle m_n, f_{m_n}(n) \rangle$, where $m_n \in \mathbb{N}$ is the unique index such that $n \in B_{m_n}$. Then g is one-to-one and therefore $\bigcup_{n \in \mathbb{N}} An \preceq \mathbb{N} \times \mathbb{N} \preceq \mathbb{N}$. □

Corollary 0.25. *The following sets are countable: $\{X \in P(\mathbb{N}) \mid X$ is finite $\}$, the set of finite sequence of natural numbers, the set of all algebraic numbers.*

Proof.

(1) Clearly $A_1 := \{X \in P(\mathbb{N}) \mid X \text{ is finite }\}$ is infinite and therefore $\mathbb{N} \preceq A_1$. To see that it is at most uncountable, note that $A_1 = \cup_{n \in \mathbb{N}} P(\{0, ..., n\})$ which is a countable union of finite (so at most countable) sets and therefore A_1 is at most countable.

(2) We are asked to prove that the set $\cup_{n \in \mathbb{N}_+} \mathbb{N}^n$ is countable. It is clearly infinite and is already given to us as a countable union of countable sets which is therefore at most countable.

(3) An algebraic number is a real number r which is a root of a non-zero polynomial with <u>integer</u> coefficients. Let $\mathbb{Z}[x]$ denote the set of all polynomials with integer coefficients. Then each non-zero polynomial has some degree $n \in \mathbb{N}$ and has the form $p(x) = z_n x^n + z_{n-1} x^{n-1} + ... z_1 x + z_0$. Let $\mathbb{Z}_n[X]$ be the set of all polynomials of degree at most n. Then clearly, $\mathbb{Z}_n[X] \sim \mathbb{Z}^{n+1}$ and therefore $\mathbb{Z}_n[X]$ is countable. Note that $\mathbb{Z}[X] = \cup_{n \in \mathbb{N}} \mathbb{Z}_n[X]$ and therefore is a countable union of countable sets (hence countable). Now the set of algebraic

numbers is just $\cup_{p(x)\in\mathbb{Z}[X]} roots(p(x))$ where $roots(p(x)) = \{r \in \mathbb{R} \mid p(r) = 0\}$. Recall that every polynomial has only finitely many roots and therefore the set of algebraic numbers is a countable union of finite sets and therefore at most countable.

$\square$

Corollary 0.26. *The following sets are uncountable:* $\{X \in P(\mathbb{N}) \mid X \sim \mathbb{N}\}$, $\mathbb{R} \setminus \mathbb{Q}$, $\{r \in \mathbb{R} \mid r \textit{ is transendental}\}$,

Proof. Lets just prove one of them. If for example $\mathbb{R}\setminus\mathbb{Q}$ was countable, then $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$ would have been a countable union of countable sets and therefore countable. Contradiction. $\square$