MATH 300: CHAPTER 4- FUNCTIONS

TOM BENHAMOU
RUTGERS UNIVERSITY

1. Functions

Relations are a wide class of important mathematical objects such as functions, orders and equivalence relation.

1.1. **Non-formal functions.** Functions are among the most common mathematical objects and appear in almost every mathematical theory. Intuitively speaking, a function is just a machine which assigns to **every** element a (the input) in a given set A (the domain of the function) a **unique** element $f(a)$ (the output/ the image of a) in a set B (the range of the function). To illustrate these ideas, here are some day-to-day examples:

(1) The function which attaches to every person its height. The domain of the function is the set of humans and the range of the functions is the set of real numbers (theoretically, a person can be 5 feet and $\sqrt{2}$ inches tall).
(2) If we attach to every person, its siblings, the result is not a function and there are two reasons for that. The first is that there are people with no siblings (and therefore the function is not defined for **every** person), also there are people with more than one sibling and for those people, we do not attach a **unique** person).

We will formally define function only later and steak with a non-formal definition for now. We will later have to justify this non-formal definition.

Definition 1.1 (Non formal)**.** Let A, B be any sets. A function from A to B is an object f, such that:

(1) f is *total* on A: for every $a \in A$, $f(a)$ is defined.
(2) f is univalent: for every $a \in A$, $f(a)$ is a unique element of B.

We denote it we $f : A \to B$. The set A is the *domain f the function f* which is denoted by $\text{dom}(f)$ and B is the *range of the function f* which we denote by $\text{rng}(f)$.

1.1.1. *How to define functions?* Usually, we declare what A and B are in advance by saying we are about to define a function $f : A \to B$. Then we provide some formula with a free variable a which we think of as a general element in the set A. This formula prescribes what element $f(a) \in B$ is assigned to a.

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Example 1.2.

(1) Define $f : \{1, 2, 3\} \to \{1, 2, 3, 4\}$ by $f(x) = x + 1$. Then $f(1) = 1 + 1 = 2$, $f(3) = 3 + 1 = 4$.

(2) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.

(3) define $f : \mathbb{R} \to \mathbb{R}$ by $f(r) = 2$, this is the constant function which for every real r returns the value 2.

(4) $f : \big\{\{1, 2, 3\}, \{1, 3, 5\}\big\} \to \mathbb{N}$, $f(X) = \max(X)$.

(5) $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(lx, y) = lx^2 + y^2, x - y + 1$.f:N→ $P(\mathbb{Z})$ $f(n) = \{n\} \cup \{1, -1\}$.

(6) Here are some non-examples:
   (a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{x}$.
   (b) $f : P(\mathbb{N}) \to \mathbb{N}$, $f(X) = \min(X)$.
   (c) $f : [0, \infty) \to [0, \infty)$, $f(x) = x - 1$.
   (d) $f : [0, \infty) \to \mathbb{R}$ $f(x) = y$ for y such that $y^2 = x$.

(8) Definition of a function by cases: Suppose we which to define a function on a set A, and for some of the elements of A we want one formula and for the another part of A we want to use a different formula. We can do that the following way: "Define $f : A \to B$ by

$$f(a) = \begin{cases} \text{(first formula)} & \text{(first condition on } a\text{)} \\ \text{(second formula)} & \text{(second condition on a)} \\ \ldots \end{cases}$$

where the conditions on a describe the element for which you would like to use the formula. When we check that a function defined by cases is well defined, we also have to check the condition on a covers all possible a and that they are "disjoint" in the sense that no a satisfy two of the condition.

   (a) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(a) = \begin{cases} \sqrt{a} & a > 0 \\ a + 1 & -1 < a \leq 0 \\ |a|^3 - 15 & a \leq -1 \end{cases}$$

   We can also use "otherwise" if we would like to take care of the remaining cases.

   (b) If we have a "small" number of elements in the domain we can use the definition by cases above to explicitly assign to every element a value, without worrying about a formula which describes that assignment. For example $f : \{1, 2, 3\} \to \{a, b, c, d\}$

$$f(x) = \begin{cases} b & x = 3 \\ a & x = 2 \\ c & x = 1 \end{cases}$$

Important: If we define $f : A \to B$ by a formula $f(a) =$(some formula) we **must** always make sure that the functions we define are well defined in the sense that:

(1) The function is total. Practically, this means that we should make sure that the formula for $f(a)$ is defined for every $a \in A$.
(2) The function is univalent. This means that for every $a \in A$, the formula for $f(a)$ points to a single element. (This is trivial in most cases)
(3) for every $a \in A$ the formula for $f(a)$ describes an element of B. So the range we declared when we wrote $f : A \to B$ is indeed correct.

Here are further examples:

(1) $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x^2$ satisfies $f(4) = 16$.
(2) $g : \mathbb{N} \to P(\mathbb{N})$ defined by $g(x) = \{x, x+1\}$ satisfies $g(5) = \{5, 6\}$.
(3) $t : \mathbb{N} \to \mathbb{N}$ defined by $t(n) = \begin{cases} 0 & n \in \mathbb{N}_{even} \\ 1 & n \in \mathbb{N}_{odd} \end{cases}$.
satisfies that $t(1) = 1, t(14) = 0$. $s(f)(3) = \{-2\}$.
(4) $F : P(\mathbb{N})^2 \to \mathbb{N}$ defined by $F(\langle A, B\rangle) = \begin{cases} 0 & A \cap B = \emptyset \\ \min(A \cap B) & else \end{cases}$
satisfies that $F(\langle \{1, 2, 3, 4\}, \mathbb{N}_{even}\rangle) = 2$.
(5) $f : \mathbb{N}^2 \to P(\mathbb{N})$ defined by $f(\langle x, y\rangle) = \{n \in \mathbb{N} \mid x < n < y\}$ satisfies $f(\langle 1, 4\rangle) = \{2, 3\}$ and $f(\langle 4, 1\rangle) = \emptyset$.

When formally working with functions we will only need the following criterion for equality of functions. This is exactly what we will have to justify once we will give the formal definition of a function:

Theorem 1.3. *Let $f, g : A \to B$ be two function. Then the following are equivalent:*

(1) $\forall x \in A. f(x) = g(x)$.
(2) $f = g$.

The theorem says that two functions with the same domain and range are equal if and only if for every x in this domain, the functions assign the same value to x. From this point, our proofs will be completely formal relaying in this theorem.

Remark 1.4. The function equality theorem indicated that a function is **not** the same as a formula defining it.

For example the functions: $f_1, f_2 : \{-1, 0, 1\} \to \mathbb{R}$ defined by $f_1(x) = |x|$ and $f_2(x) = x^2$ have different formulas but they define the same function since $f_1(-1) = f_2(-1)$, $f_1(0) = f_2(0)$, $f_1(1) = f_2(1)$.

Remember! Different formulas can define the same function.

1.1.2. *Operations on functions.*

Definition 1.5. Let $f : A \to B$ be a function and $X \subseteq A$. We define the *restriction of f to X*, denoted by $f \restriction X : X \to B$, to be the function with domain $dom(f \restriction X) = X$ and for every $x \in X$, $(f \restriction X)(x) = f(x)$.

Intuitively, the restriction of a function acts the same way that the original function did, the only difference is that the domain restricts to the new set X.

Definition 1.6. Let A be any set. We define the *Identity function* on A as the function $Id_A : A \to A$ defined by $Id_A(a) = a$.

Example 1.7. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined as $f(z) = |z|$. Prove that $f \restriction \mathbb{N} = Id_{\mathbb{N}}$

Proof. We want to prove equality of functions. First we want to prove that $\mathrm{dom}(f \restriction \mathbb{N}) = \mathrm{dom}(Id_{\mathbb{N}})$. Indeed by definition of restriction and the identity function, both of the functions have domain $\mathbb{N}$. Next we want to prove that $\forall x \in \mathbb{N}.(f \restriction \mathbb{N})(x) = Id_{\mathbb{N}}(x)$. Let $x \in \mathbb{N}$, then by definition of restriction and since $n \geq 0$ we have

$$(f \restriction \mathbb{N})(x) = f(x) = |x| = x$$

and by definition of the identity function we have

$$Id_{\mathbb{N}}(x) = x$$

Hence

$$(f \restriction \mathbb{N})(x) = x = Id_{\mathbb{N}}(x)$$

as wanted $\square$

Definition 1.8. Let $f : A \to B$ and $g : B \to C$ be two functions. We define the composition of g in f as $g \circ f : A \to C$, to be the function with domain f and range C such that for each $a \in A$, $(g \circ f)(a) = g(f(a))$.

Example 1.9.
(1) $f(x) = x^2$ and $g(x) = x + 1$, then $g \circ f(x) = x^2 + 1$ and $f \circ g(x) = (x + 1)^2$.
(2) $f : P(\mathbb{N}) \setminus \{\emptyset\} \to \mathbb{N} \times \mathbb{N}$, $f(X) = \langle \min(X), \min(X) + 1\rangle$ and $g : P(\mathbb{N}) \to P(\mathbb{N}) \setminus \{\emptyset\}$, $g(X) = X \cup \{0\}$. Then $f \circ g(X) = f(X \cup \{0\}) = \langle \min(X \cup \{0\}), \min(X \cup \{0\}) + 1\rangle = \langle 0, 1\rangle$.

Proposition 1.10. *Suppose that $f : A \to B$, $g : B \to C$ and $h : C \to D$. Then:*

(1) $f \circ Id_A = f$, $Id_B \circ f = f$.
(2) $h \circ (g \circ f) = (h \circ g) \circ f$.

Proof. Let us prove for example that $f \circ Id_A = f$. We need to prove function equality, the domain of both functions is A. Let $a \in A$, then $(f \circ Id_A)(a) = f(Id_A(a)) = f(a)$ hence $f \circ Id_A = f$. $\square$

1.1.3. *Properties of functions.*

Definition 1.11. Let $f : A \to B$ be a function we sat that f is:

(1) One to one/ injective: if for every $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$ then $a_1 = a_2$.
(2) Onto/ surjective: if for every $b \in B$ there is $a \in A$ such that $f(a) = b$.

Example 1.12. (1) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not injective as $1 \neq -1$ and $f(-1) = (-1)^2 = 1 = 1^2 = f(1)$.

(2) $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(n) = n - 1$ is injective.

Proof. Let $n_1, n_2 \in \mathbb{N}$. Suppose that $f(n_1) = f(n_2)$, we want to prove that $n_1 = n_2$. By definition of f, $n_1 - 1 = n_2 - 1$, adding 1 to both sides of the equation we conclude that $n_1 = n_2$. □

(3) $g : \mathbb{N}\times\mathbb{N} \to \mathbb{N}\times\mathbb{N}$ defined by $g(\langle n, m\rangle) = \langle 2n+m, n+m\rangle$ is injective.

Proof. Let $\langle n_1, m_1\rangle, \langle n_2, m_2\rangle \in \mathbb{N}\times\mathbb{N}$ and assume that $g(\langle n_1, m_1\rangle) = g(\langle n_2, m_2\rangle)$ we want to prove that $\langle n_1, m_1\rangle = \langle n_2, m_2\rangle$. By the assumption we know that $\langle 2n_1+m_1, n_1+m_1\rangle = \langle 2n_2+m_2, n_2+m_2\rangle$ and by equality of pair we get that

$$2n_1 + m_1 = 2n_2 + m_2 \text{ and } n_1 + m_1 = n_2 + m_2$$

Subtracting the second equation from the first we get:

$$2n_1 + m_1 - (n_1 + m_1) = 2n_2 + m_2 - (n_2 - m_2)$$

$$n_1 = n_2$$

Hence by the equality $n_1 + m_1 = n_2 + m_2$, we have that $n_1 = n_2$ cancels so $m_1 = m_2$. By equality of pairs we conclude that $\langle n_1, m_1\rangle = \langle n_2, m_2\rangle$. □

(4) $F : P(\mathbb{N}) \to P(\mathbb{N})$ defined by $F(X) = \{x + 1 \mid x \in X\}$ is injective.

Proof. Let $X_1, X_2 \in P(\mathbb{N})$, suppose that $F(X_1) = F(X_2)$ we want to prove that $X_1 = X_2$. By definition of F,

$$)*) \quad \{x + 1 \mid x \in X_1\} = \{x + 1 \mid x \in X_2\}$$

Let us prove $X_1 = X_2$ by a double inclusion:

(a) $\underline{X_1 \subseteq X_2}$: Let $x_0 \in X_1$ we want to prove that $x_0 \in X_2$. By definition $x_0+1 \in \{x+1 \mid x \in X_1\}$ and by $(*)$, $x_0+1 \in \{x+1 \mid x \in X_2\}$. By the replacement principle, there exists $y \in X_2$ such that $x_0+1 = y+1$, hence $x_0 = y \in X_2$, which implies that $x_0 \in X_2$ as wanted.

(b) $\underline{X_2 \subseteq X_1}$: Symmetric to the first inclusion.

□

(5) $F_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $F(\langle n, m\rangle) = 2^n \cdot 3^m$ is injective.

Proof. Let $\langle n_1, m_1\rangle, \langle n_2, m_2\rangle \in \mathbb{N} \times \mathbb{N}$. Suppose that $F_1(n_1, m_1) = F_1(n_2, m_2)$ we want to prove that $\langle n_1, m_1\rangle = \langle n_2, m_2\rangle$. By definition of F_2 we have that $(*)$ $2^{n_1}3^{m_1} = 2^{n_2}3^{m_2}$. By the fundamental theorem of arithmentics, each positive natural number has a unique factorization into primes. The equality $(*)$ provides two factorization into primes of the same numbers, hence it must be the same, namely $n_1 = n_2$ and $m_1 = m_2$. By the basic property of pairs, $\langle n_1, m_1\rangle = \langle n_2, m_2\rangle$. □

Definition 1.13. Let $f : A \to B$ be a function. The image of f, denoted by $Im(f) = \{f(x) \mid x \in A\}$.

Exercise. *For the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^2$ Prove that* $\text{dom}(f) = Rng(f) = \mathbb{R}$ *while* $\text{Im}(f) = [0, \infty)$.

Since the last equality if a set equality, we should prove it by a double implication:

(1) $\subseteq$: Let $r \in \text{Im}(f)$, we need to prove that $r \in [0, \infty)$. By definition of $\text{Im}(f)$, there is $x \in \mathbb{R}$ such that $f(x) = r$. Those $r = x^2 \geq 0$ and by definition of $[0, \infty)$, $r \in [0, \infty)$.
(2) $\supseteq$: Let $r \in [0, \infty)$. we need to prove that $r \in \text{Im}(f)$. By definition, $r \geq 0$ and therefore we have $\sqrt{r}$ defined. Define (This is an existential proof) $x = \sqrt{r}$, then $f(x) = x^2 = r$.

Remark 1.14. f is surjective if and only if $Im(f) = Range(f)$.

Example 1.15. (1) The function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 2n$ is not surjective.

Proof. For example $1 \in \mathbb{N}$ and for every $n \in \mathbb{N}$, $f(n) \neq 1$. Otherwise, there exists $n \in \mathbb{N}$ such that $f(n) = 1$ then by definition of f, $2n = 1$ which implies that 1 is even, contradiction. □

Note also that $Im(f) = \mathbb{N}_{even}$ and that f is injective.

(2) The function $g : P(\mathbb{Z}) \to P(\mathbb{N})$ defined by $g(X) = X \cap \mathbb{N}$ is surjective.

Proof. Let $Y \in P(\mathbb{N})$ we want to prove that there is $X \in P(\mathbb{Z})$ such that $f(X) = Y$. Define $X = Y$, then since $Y \in P(\mathbb{N})$, $Y \in P(\mathbb{Z})$. Also, to see that $g(Y) = Y$, we need to prove that $Y \cap \mathbb{N} = Y$. This is equivalent (by a proposition we have seen previously) to the fact that $Y \subseteq \mathbb{N}$. This follows since $Y \subseteq \mathbb{N}$. □

Also note that $Im(g) = P(\mathbb{N})$, (since we just proved that g is surjective) and it is not injective since for example $g(\{-1, 1\}) = \{1\} = g(\{1\})$.

(3) The function $h : (0, \infty) \to (0, \infty)$ defined by $h(x) = \frac{1}{x}$ is surjective.

Proof. Let $y \in (0, \infty)$, we want to prove that there is $x \in (0, \infty)$ such that $h(x) = y$. Namely, we want that $\frac{1}{x} = y$. Then define $x = \frac{1}{y}$. Since $0 < y$, also $0 < x$ and therefore $x \in (0, \infty)$ and we have that $h(x) = \frac{1}{\frac{1}{y}} = y$ as wanted. □

(4) $G : P(\mathbb{N}) \times P(\mathbb{N}) \to P(\mathbb{N} \times \mathbb{N})$ defined by $G(\langle X, Y \rangle) = X \times Y$ is not onto.

Proof. For example $\{\langle 1, 1\rangle, \langle 2, 2\rangle\} \in Range(G) \setminus Im(G)$. Suppose toward a contradiction that $G(\langle X, Y \rangle) = \{\langle 1, 1\rangle, \langle 2, 2\rangle\}$. Then by definition of G, $X \times Y = \{\langle 1, 1\rangle, \langle 2, 2\rangle\}$. By set equality, this means that $\langle 1, 1\rangle$, $\langle 2, 2\rangle \in X \times Y$. which by the definition of Cartesian product implies that $1, 2 \in X$ and $1, 2 \in Y$. But then $\langle 1, 2\rangle \in X \times Y$ but $\langle 1, 2\rangle \notin \{\langle 1, 1\rangle, \langle 2, 2\rangle\}$, contradiction. □

Proposition 1.16. *Let $f : A \to B$ and $g : B \to C$ be any functions.*

(1) If f, g are injective then so is $g \circ f$.

(2) If f, g are surjective then so is $g \circ f$

Definition 1.17. A function $f : A \to B$ is invertible if there is a function $g : B \to A$ such that:

$$g \circ f = id_A \quad \text{and } f \circ g = id_B$$

Example 1.18. (1) $f : \{a, b, c\} \to \{1, 2, 3\}$ defined by

$$f(x) = \begin{cases} 1 & x = a \\ 2 & x = b \\ 3 & x = c \end{cases}$$

is invertible as witnessed by the function $g : \{1, 2, 3\} \to \{a, b, c\}$,

$$g(x) = \begin{cases} a & x = 1 \\ b & x = 2 \\ c & x = 3 \end{cases}$$

(2) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + 1$ is invertible since the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x - 1$ satisfy that $g \circ f = f \circ g = Id_{\mathbb{R}}$.
(3) The function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n + 1$ is **not** invertible. The function $g(n) = n-1$ is not a function from $\mathbb{N}$ to $\mathbb{N}$ as $g(0) = -1$. To formal way to prove it is to use the next theorem (and the fact the g is not onto). If we restrict the range of f to $\mathbb{N}_+$ then g above from $\mathbb{N}_+$ to $\mathbb{N}$ witnesses that f is invertible.
(4) There is no $f : \{a, b, c\} \to \{1, 2, 3, 4\}$ which is invertible.
(5) $f : P(\mathbb{N}) \to P(\mathbb{N})$ defined by $f(X) = \mathbb{N} \setminus X$ is invertible as $f \circ f = Id_{P(\mathbb{N})}$.

Theorem 1.19. *If g_1, g_2 are two inverse functions of f then $g_1 = g_2$. We denote the inverse function of f by f^{-1}.*

Proof. Suppose the g_1, g_2 are two inverse function of f, then

$$g_1 \circ f = id_A \quad \text{and } f \circ g_1 = id_B$$

$$g_2 \circ f = id_A \quad \text{and } f \circ g_2 = id_B$$

It follows that

$$g_1 = g_1 \circ Id_B = g_1 \circ (f \circ g_2) = (g_1 \circ f) \circ g_2 = Id_A \circ g_2 = g_2$$

□

Theorem 1.20. *A function $f : A \to B$ is invertible if and only if it is one to one and onto.*

Proof. Suppose that f is invertible and let $f^{-1} : B \to A$ be the inverse function. Let us prove that f is one to one and onto:

- one to one: Let $a_1, a_2 \in A$, suppose that $f(a_1) = f(a_2)$, we want to prove that $a_1 = a_2$. Then $f^{-1}(f(a_1)) = f^{-1}(f(a_2))$ and since $f^{-1} \circ f = Id_A$ we get that
$$a_1 = f^{-1}(f(a_1)) = f^{-1}(f(a_2)) = a_2$$
- onto: Let $b \in B$, we want to prove that there is $a \in A$ such that $f(a) = b$. Let $a = f^{-1}(b) \in A$. Then $f(a) = f(f^{-1}(b))$ and since $f \circ f^{-1} = Id_B$, we have that $f(a) = f(f^{-1}(b)) = b$ as wanted.

For the other direction, suppose that f is one to one and onto B. We want to prove that f is invertible, namely that there is a function $g : B \to A$ such that $f \circ g = Id_B$ and $g \circ f = Id_A$. Here is the definition of g: For any element of b, there is (since f is onto B) a unique (since f is one to one) element $a_b \in A$ such that $f(a_b) = b$. Define $g(b) = a_b$. Let us prove that g is inverse to f:

- $g \circ f = Id_A$: Let $a \in A$, then denote $f(a) = b \in B$. By definition $g(b) = a_b$ is the unique element in A such that $f(a_b) = b$ and since $f(a) = b$ it follows that $a = a_b$. Hence $g(f(a)) = g(b) = a_b = a$. It follows that $g \circ f = Id_A$.
- $f \circ g = Id_B$: Let $b \in B$, by definition, $g(b) = a_b$ and a_b has the property that it is (the unique which is) mapped to b, namely $f(a_b) = b$. Hence $f(g(b)) = f(a_b) = b$. Again it follows that $f \circ g = Id_B$.

$\square$

1.2. **General relations.** Toward a formal definition of a function, we would like to describe that certain objects relate to other objects. To turn relations into a formal mathematical object, we need to define them as sets. First, how would we code that an object a relates to an object b? we can use the ordered pair $la, b. A single relation describes many such connections, hence it is a set of ordered pairs:$

Definition 1.21. A relation from the set A to the set B is set $R \subseteq A \times B$.

Example 1.22. (1) $R = \{\langle 1, 2\rangle, \langle 1, 3\rangle\}$ is a relation from $\{1, 2\}$ tp $\{1, 2, 3\}$ since
$$R \subseteq \{1, 2\} \times \{1, 2, 3\}$$
. R is also a relation from $\mathbb{R}$ to $\mathbb{N}$.
(2) $\{\langle 1, \sqrt{2}\rangle, \langle 2, 4\rangle\}$ is not a relation from $\mathbb{N}$ to $\mathbb{N}$.
(3)
$$id_{\mathbb{N}} = \{\langle n, n\rangle \mid n \in \mathbb{N}\}$$
$\leq_{\mathbb{N}} = \{\langle n, m\rangle \in \mathbb{N}^2 \mid \exists k \in \mathbb{N}. n+k = m\}$, $<_{\mathbb{N}} = \{\langle n, m\rangle \in \mathbb{N}^2 \mid \exists k \in \mathbb{N}_+. n+k = m\}$
are three relations from $\mathbb{N}$ to $\mathbb{N}$. Note that
$$\leq = < \cup id_{\mathbb{N}}$$
(4) $A = \{\langle x, y\rangle \in \mathbb{R}^2 \mid x - y \in \mathbb{Q}\}$ for example $\langle 3 + \sqrt{2}, \sqrt{2}\rangle \in A$, $\langle 1, \pi\rangle \notin A$.
(5) $R = \{\langle X, Y\rangle \in P(\mathbb{N}) \times P(\mathbb{Z}) \mid X \subseteq Y\}$. R is a relation from $P(\mathbb{N})$ to $P(\mathbb{Z})$.

(6) It is sometimes convinient to imagine a relation as two potato's representing the sets A and B, and then and arrows from A to B. For example, if $R = \{\langle 1,2\rangle, \langle 2,a\rangle, \langle 2,b\rangle\}$ From $\{1,2,3\}$, to $\{2,a,b\}$:

relation.png

(7) $S = \{lx, y$

Remark 1.23. In most cases a relation (i.e. a set of pairs) has a "meaning", which is some notion we already familiar with, just not in terms of sets of pairs. In the previous examples, $\leq_{\mathbb{N}}$ is just a formal representation for the usual $\leq$ where we only consider natural numbers. The relation D is just the divisibility relation on between integers, and id_A is just the equality relation where we only consider elements of the set A. **However**, a general relation R, is just an abstract object. It does not necessarily have a meaning as in the previous examples. Examples $(1), (2), (6)$ do not arise from a natural notion. We can always artificially force a meaning to it, but this would be of no use.

Definition 1.24. Let R be a relation from A to B. Define:

Definition 1.25. (1) $\text{dom}(R) = \{a \in A \mid \exists b \in B,\ la, b$

<u>**Important**</u>: When handling general relations, do not try to find a "meaning" for it. Instead, you should simply think of a set of pairs. When handling a specific relation, it is important to understand the idea behind it (by finding examples pairs of elements which belongs to the relation).

Problem 1. *Let R be a relation from A to B, S be a relation from B to C. Define*

$$S \circ R = \{la, c$$

1.3. **abstract functions.** The formal way to define a function is as relations:

Definition 1.26. Let A, B be two sets. A *function* from A to B is a relation from A to B such that:

(1) f is total *Total* on A: $\forall a \in A.\exists b \in B.la, b$

Notation 1.27. If f is a function from A to B we denote it by $f : A \to B$. Also if $f : A \to B$ is a function, we denote $f(a) = b$ if and only if la, b

Example 1.28. (1) Let $f = \{\langle 1,a\rangle, \langle 3,b\rangle, \langle 2,a\rangle\}$. To see that f is a function from $\{1,2,3\}$ to $\{a,b,c\}$, we need to prove that for every $x \in \{1,2,3\}$ the is a unique $y \in \{a,b,c\}$ such that lx, y

Proof. (2) We need to prove that f_b is total on A and univalent.
<u>Total:</u> We need to prove that for every $x \in A$ there is $y \in B$ such that lx, y
Hence $f_b : A \to B$ is a function satisfying $\forall a \in A.f_b(a) = b$.

$\pi_1 : A \times B \to A$ $\pi_1 = \{\langle\langle a, b\rangle, c\rangle \in (A \times B) \times A \mid a = c\}$ Is called the *projection to the left coordinate*, it satisfies that $\pi(la, b\mathring{)} = a$. Similarly, the *projection to the right coordinate* is denoted $\pi_2 : (A \times B) \to B$ and it satisfies $\pi_2(\langle a, b\rangle) = b$.
To summation operation on the rational number (or on the natural numbers/integers/reals) is a function $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. We are used to write $3 + 5 = 8$ instead of $+(l3, 5\mathring{)} = 8$.
Let $g : P(A) \times P(B) \to P(A)$ defined by $g = \{\langle\langle X, Y\rangle, Z\rangle \in (P(A) \times P(B)) \times P(A) \mid Z = X \cap Y\}$ we have that $g(X, Y) = X \cap Y$
Given a set of pairs R in $A \times B$ we can represent R as a collection of arrows from he set A to the set B. This is very convenient when considering functions. For example, to verify the R is a function from A to B we should simply verify(not prove!) that there is **exactly one** arrow attached to **every** element of A. For example, consider

$$f : \{1, 2, 3, 4\} \to \{-1, 0, 1, 2, 3, 4, 5\} \quad f = \{\langle 1, 1\rangle, \langle 2, 1\rangle, \langle 3, 3\rangle\langle 4, 5\rangle\}$$

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Definition 1.29. A sequence of elements in the set A is a function $f : \mathbb{N} \to A$. In calculus we sometime denote $a_n = f(n)$ and $(a_n)_{n=0}^{\infty} = f$.

Example 1.30. The sequence $1, \frac{1}{2}, \frac{1}{3}, ...$ is formally the function $f : \mathbb{N} \to \mathbb{Q}$, $f = \{ln, \frac{1}{n+1}$

Definition 1.31. Let $f : A \to B$ be a function. The *domain* of f is simply A, we denote $\mathrm{dom}(f) = A$. The *range* of f is B and we denote $\mathrm{rng}(f) = B$. The *image* of f is the set $\mathrm{Im}(f) = \{f(a) \mid a \in A\}$.

Definition 1.32. Let A, B be two sets. We denote *the set of all functions* from A to B by

$$^{A}B = \{f \in P(A \times B) \mid f \text{ is a function from } A \text{ to } B\}$$

Example 1.33. Let F_2 be the relation from $^{\mathbb{R}}\mathbb{R}$ to $\mathbb{R}$ defined by

$$F_2 = \{\langle f, r\rangle \in {}^{\mathbb{R}}\mathbb{R} \times \mathbb{R} \mid \langle 2, r\rangle \in f\}.$$

Prove that F is a function.

Proof. Total: We nee to probe that for every $f \in \mathbb{R}\mathbb{R}$ (here the domain of F_2 is itself a set of functions!) there is $r \in \mathbb{R}$ such that lf, r Note that we have $F_2(f) = f(2)$ for every function $f \in {}^{\mathbb{R}}\mathbb{R}$.

In order to discard the need to formulate functions as sets of pair we simply need to understand when two functions are equal[2].

Theorem 1.34. *Let f, g be any function. Then the following are equivalent:*

[2] As we did with tuples.

(1) $f = g$ *(equality of sets of pairs!).*
(2) $\mathrm{dom}(f) = \mathrm{dom}(g)$ *and for every* $x \in \mathrm{dom}(f)$, $f(x) = g(x)$.

Proof. $\underline{\Rightarrow:}$ Suppose that $f = g$, then clearly $\mathrm{dom}(f) = \mathrm{dom}(g)$. Let $x \in \mathrm{dom}(f)$, and denote by $f(x) = y$. Then lx, y

$\underline{\Leftarrow:}$ $\underline{\subseteq:}$ Let lx, y

□

Problem 2. *Let* $f : A \to B$ *be a function.*

(1) *Prove that if* $X \subseteq A$*, then* $f \cap X \times B$ *is a function and equals* $f \restriction X$.
(2) *Show that if* $f : A \to B$, $g : B \to C$ *are functions then* $g \circ f$ *(the composition of the relations) is a function from* A *to* C *and that for every* $a \in A$, $g \circ f(a) = g(f(a))$.
(3) *Prove that if* f *is one-to-one and onto* B *then* f^{-1} *(the inverse relation) is a function and moreover that* $f^{-1} \circ f = Id_A$ *and* $f \circ f^{-1} = Id_B$.