Introductory Linear Algebra-Midterm I Solutions

MATH 250 (Instructor: Tom Benhamou) September 30, 2024

Instruction

The midterm consists of 3 problems, each worth 34 points (The maximal grade is 100). For this you will have one hour. No material is allowed. The solutions to the problems should be written in the designated areas and the “extra page" at the end.

Full Name (PRINT): ______________________________

Net ID: ______________________________

Introductory Linear Algebra-Midterm I Solutions

MATH 250 (Instructor: Tom Benhamou) September 30, 2024

Problems

Problem 1. For each of the following statements determine if it is true are false. Provide a counterexample if false. No explanation is required if true (circle the correct answer):

a. If A is a 2×3 matrix then the equation $A \cdot \bar{X} = \bar{0}$ has infinitely many solutions. True \ False

explanation(not needed in the exam): It is homogeneous so it is consistent, and in any Echelon form there will be a column without a leasing entry so by the algorithm we saw in class, there are infinitely many solutions.

b. If A is a 2×3 matrix and $\bar{C} \in \mathbb{R}^3$, then whenever $A \cdot \bar{C} = \bar{0}$, either $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ or $\bar{C} = \bar{0}$. True \ False

counter example: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\bar{C} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

c. If the columns of a 4×3 matrix A are linearly independent then for any $\bar{b} \in \mathbb{R}^4$, the equation $A \cdot \bar{X} = \bar{b}$ have a unique solution. True \ False

counter example: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\bar{b} = \bar{e}_4$

(note that $\bar{b}$ is not needed. We only need the exmaple of A.)

Introductory Linear Algebra-Midterm I Solutions

MATH 250 (Instructor: Tom Benhamou) September 30, 2024

Problem 2. Find specific values of (a,b,c) such that $\begin{bmatrix}1\\a\\1\end{bmatrix},\begin{bmatrix}-1\\b\\-1\end{bmatrix},\begin{bmatrix}c\\0\\1\end{bmatrix}$ is linearly independent, and find values so it is linearly dependent. Explain your answer.

Solution: The vectors $\begin{bmatrix}1\\a\\1\end{bmatrix},\begin{bmatrix}-1\\b\\-1\end{bmatrix},\begin{bmatrix}c\\0\\1\end{bmatrix}$ are lineraly independent if and only if the vector equation:

$$x\begin{bmatrix}1\\a\\1\end{bmatrix}+y\begin{bmatrix}-1\\b\\-1\end{bmatrix}+z\begin{bmatrix}c\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

has a unique solution (i.e. the trivial solution). To check this let us eliminate the corresponding matrix

$$\begin{bmatrix}1 & -1 & c\\ a & b & 0\\ 1 & -1 & 1\end{bmatrix}\overset{\substack{R_2\to R_2-aR_1\\ R_3\to R_3-R_1}}{\longrightarrow}\begin{bmatrix}1 & -1 & c\\ 0 & b+a & -ca\\ 0 & 0 & 1-c\end{bmatrix}$$

Since we are only interested in the number of solutions we do not need to get to the reduced Echelon form. For $(a,b,c)=(0,1,1)$, we get

$$\begin{bmatrix}1 & -1 & 1\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

since in the third column there is no leading entry there are non-trivial solutions, in which case the sequence is <u>linearly dependent</u>.

For $(a,b,c)=(0,1,0)$ we get

$$\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

In which case there is a unique solution and therefore the sequence is linearly independent.

Problem 3. Determine whether the sequence $\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \bar{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \bar{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ is spanning. If yes, justify it, if not, give an example of $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \notin \mathrm{Sp}(\bar{v}_1, \bar{v}_2, \bar{v}_3)$

solution: As we have seen in class, to check whether the sequence is spanning we should put the vectors as columns of a matrix, eliminate, and check whether in the Echelon form there is a 0-row.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

we see that there is a 0-row, hence the sequence is not spanning. To find $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \notin \mathrm{Sp}(\bar{v}_1, \bar{v}_2 \bar{v}_3)$, we need to find a triple which is not a linear combination of $\bar{v}_1, \bar{v}_2, \bar{v}_3$. Do to so, we can eliminate the matrix

$$\begin{bmatrix} 1 & 2 & 3 & |a \\ 1 & 1 & 1 & |b \\ 1 & 1 & 1 & |c \end{bmatrix}$$

which corresponds to checking whether $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a linear combination of $\bar{v}_1, \bar{v}_2, \bar{v}_3$, and find a, b, c for which the corresponding equation has no solutions

$$\begin{bmatrix} 1 & 2 & 3 & |a \\ 1 & 1 & 1 & |b \\ 1 & 1 & 1 & |c \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & 3 & |a \\ 1 & 1 & 1|b & \\ 0 & 0 & 0|c-b & \end{bmatrix}$$

So taking for example $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, produces a contradiction row in the third row.

Extra Page: