Introductory Linear Algebra-Midterm III Preparation questions

MATH 250 (Instructor: Tom Benhamou) November 18, 2024

Instruction

The structure and instructions for Midterm III is identical to Midterm I.

Problems

Problem 1. Anwer the following questions. No explanation is required:

a. $\text{rank}(\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 8 & 1 \end{bmatrix}) =$ _____2_____

b. There is a 6×4 matrix A with $\text{nullity}(A) = 2$ and $\text{rank}(A) = 2$

False \ True and give an example: $\underline{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$

c. $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$. True \ <u>False</u>

d. $\{\begin{bmatrix} x \\ y+1 \\ z \end{bmatrix} \mid x, y, z \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$. <u>True</u> \ False.

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e. For any bases $\mathcal{B}, \mathcal{C}$ of a vector space V, the matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is invertible.

<u>True</u> \ False

counter example: ______________________________

f. For any $m \times n$-matrix A of rank k, erasing the last column results in a matrix of rank $k-1$. True \ <u>False</u>

counter example: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has rank 2 and if we erase the last column it still has rank 2.

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Problem 2. Suppose that $S : \mathbb{R}^3 \to \mathbb{R}^2$ is given by $S(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x - z \\ z - 2y - x \end{bmatrix}$.

Find bases for Ker(S) and Im(S).

Solution:

The standard matrix of S is $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & -2 & 1 \end{bmatrix}$ By a theorem we saw in class it suffices to find bases for $null(A) = Ker(S)$ and $Cols(A) = Im(S)$. We eliminate A to

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

So the pivot columns are 1,2 and therefore a basis for $cols(A) = Im(S)$ is given by $\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \end{bmatrix}\}$. Also, the system of the reduces form is

$$x - z = 0, \; y = 0$$

So a general element of $null(A)$ has the form

$$\begin{bmatrix} z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

hence $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ form a basis for $null(A)$.

Problem 3. Find all the eigenvalues of the matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. For each eigenvalue of A present the eigenspace as the span of vectors.

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Problem 4. Let $T : \mathbb{P}_2(\mathbb{R}) \to \mathbb{P}_2(\mathbb{R})$ be the map $T(p) = p + p'$, where p' is the derivative of p. Is T an isomorphism?

Problem 5. Let H be the subspace of $M_{3\times 3}(\mathbb{C})$ consisting of all matrices A with $A_{11} + A_{22} + A_{33} = 0$.

- Prove that H is a subspace of $M_{3\times 3}(\mathbb{C})$.
- What is $\dim(H)$?

Problem 6. Let $\mathcal{B} = \{\begin{bmatrix} 1 \\ i \\ 1+i \end{bmatrix}, \begin{bmatrix} 1+i \\ 2 \\ -1-i \end{bmatrix}, \begin{bmatrix} 1+2i \\ 3-i \\ -i \end{bmatrix}\}$, show that $\mathcal{B}$ is a basis for $\mathbb{C}^3$ and compute $[\begin{bmatrix} 2-i \\ -1+3i \\ 1+2i \end{bmatrix}]_{\mathcal{B}}$.

Problem 7. Prove that if $\bar{x}$ is an eigenvector for A, then $\bar{x}$ is an eigenvector for A^2.

Proof. Suppose that $\bar{x}$ is an eigen vector of A, then there is λ such that $A\bar{x} = \lambda\bar{x}$. It follows that

$$A^2\bar{x} = A(A\bar{x}) = A(\lambda\bar{x}) = \lambda(A\bar{x}) = \lambda^2\bar{x}$$

Hence $\bar{x}$ is an eigenvector for A^2 for the eigenvalue λ^2. □

Problem 8. Suppose that $\mathcal{B} = \{\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4\}$ is a basis for V and $\mathcal{C} = \{\bar{b}_1 + \bar{b}_1, \bar{b}_1 + \bar{b}_2, \bar{b}_1 + \bar{b}_3, \bar{b}_1 + \bar{b}_4\}$. Show that $\mathcal{C}$ is also a basis for V and compute $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$.

Proof. The first part is a problem we did in class, let us reproduce the solution. Since $\mathcal{B}$ is a basis for V, $\dim(V) = 4$ and since C has 4 vectors, by a theorem we saw in class it suffices to check that C is LI. Suppose that

$$x_1(\bar{b}_1 + \bar{b}_1) + x_2\bar{b}_1 + \bar{b}_2) + x_3(\bar{b}_1 + \bar{b}_3) + x_4(\bar{b}_1 + \bar{b}_4) = \bar{0}$$

We need to show that $x_1 = x_2 = x_3 = x_4 = 0$. Rearranging the above equations we get

$$(2x_1 + x_2 + x_3 + x_4)\bar{b}_1 + x_2\bar{b}_2 + x_3\bar{b}_3 + x_4\bar{b}_4 = \bar{0}$$

Since $\mathcal{B}$ is LI, $2x_1 + x_2 + x_3 + x_4 = 0$, $x_2 = x_3 = x_4 = 0$. Hence $2x_1 = 0$ and therefore $x_1 = 0$.

For the second part,

$$\underset{\mathcal{B}\leftarrow C}{P} = \begin{bmatrix} | & | & | & | \\ [\bar{b}_1 + \bar{b}_1]_{\mathcal{B}} & [\bar{b}_1 + \bar{b}_2]_{\mathcal{B}} & [\bar{b}_1 + \bar{b}_3]_{\mathcal{B}} & [\bar{b}_1 + \bar{b}_4]_{\mathcal{B}} \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

□

Problem 9. Let $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ and $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ be two LI sequences of vectors in V.

(a) Is $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{u}_1, \bar{u}_2, \bar{u}_3\}$ LI?

Solution. No for example $\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}$ and $\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\}$ are both LI but all of them together are not (since more than 3 vectors in $\mathbb{R}^3$ are linearly dependent.)

(b) Show that if the only vector in $Sp(\{\bar{v}_1, \bar{v}_2, \bar{v}_3\})$ and in $Sp(\{\bar{u}_1, \bar{u}_2, \bar{u}_3\})$ is the zero vector then $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{u}_1, \bar{u}_2, \bar{u}_3\}$ is LI.

Proof. Let $x_1\bar{v}_1 + x_2\bar{v}_2 + x_3\bar{v}_3 + y_1\bar{u}_1 + y_2\bar{u}_2 + y_3\bar{u}_3 = 0$ we need to show that $x_1 = x_2 = x_3 = y_1 = y_2 = y_3 = 0$. We have that

$$\bar{w} = x_1\bar{v}_1 + x_2\bar{v}_2 + x_3\bar{v}_3 = -y_1\bar{u}_1 - y_2\bar{u}_2 - y_3\bar{u}_3$$

Hence $\bar{w}$ is a linear combination of both sequences which implies that $\bar{w}$ is in both $Sp(\{\bar{v}_1, \bar{v}_2, \bar{v}_3\})$ and in $Sp(\{\bar{u}_1, \bar{u}_2, \bar{u}_3\})$. By our assumption, this means that $\bar{w} = 0$, so

$$\bar{w} = x_1\bar{v}_1 + x_2\bar{v}_2 + x_3\bar{v}_3 = -y_1\bar{u}_1 - y_2\bar{u}_2 - y_3\bar{u}_3 = 0$$

Since $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ and $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ are LI, then $x_1 = x_2 = x_3 = 0$ and $-y_1 = -y_2 = -y_3 = 0$ which implies $y_1 = y_2 = y_3 = 0$. □