Introductory Linear Algebra-Midterm II
Preparation questions

MATH 250 (Instructor: Tom Benhamou) September 19, 2024

Instruction

The structure and instructions for Midterm II is identical to Midterm I.

Problems

Problem 1. For each of the following statements determine if it is true are false. Provide a counter example if false. No explanation is required if true (circle the correct answer):

a. $T : \mathbb{R}^2 \to \mathbb{R}^2$ define by $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x + 2y \\ 2xy \end{bmatrix}$ is linear. True \ False

counter example: $T(\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix}) = T(\begin{bmatrix} 5 \\ 5 \end{bmatrix}) = \begin{bmatrix} 15 \\ 50 \end{bmatrix} \neq T(\begin{bmatrix} 2 \\ 2 \end{bmatrix}) + T(\begin{bmatrix} 3 \\ 3 \end{bmatrix}) = \begin{bmatrix} 6 \\ 8 \end{bmatrix} + \begin{bmatrix} 9 \\ 18 \end{bmatrix}$.

b. Any linear map $T : \mathbb{R}^3 \to \mathbb{R}^4$ such that $\{T(\bar{e}_1), T(\bar{e}_2), T(\bar{e}_3)\}$ are linearly independent must be onto. True \ False

counter example: $T(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$. Indeed $\{T(e_1), T(e_2), T(e_3)\} = \{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\}$

but T is not onto since for example $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is not in the range of T.

Introductory Linear Algebra-Midterm II Preparation questions

MATH 250 (Instructor: Tom Benhamou) September 19, 2024

c. Any linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\{T(\bar{e}_1), T(\bar{e}_2), T(\bar{e}_3)\}$ is spanning $\mathbb{R}^3$, must also be one-to-one. True \ False

explanation 1: $\{T(\bar{e}_1), T(\bar{e}_2), T(\bar{e}_3)\}$ are the columns of the 3×3 standard matrix A for the transformation T. So the columns of A are spanning, By a theorem we saw in class, any square matrix whose columns are spanning is invertible. Hence A is invertible, and by a theorem we saw in class, T is invertible. By yet another theorem from class, T being invertible implies that T T is one-to-one.

explanation 2: By a theorem from class, if $\{T(\bar{e}_1), T(\bar{e}_2), T(\bar{e}_3)\}$ is spanning then T is onto. Therefore $T : \mathbb{R}^3 \to \mathbb{R}^3$ is an onto linear transformation. By the theorem we saw in class T is invertible and therefore (by the theorem we stated in exp. 1) T is one-to-one

d. $\det(\alpha \cdot A) = \alpha \cdot \det(A)$ for any square matrix A and any scalar α. True \ False

counter example: $\det(2 \cdot I_2) = 4 \neq 2 = 2 \cdot \det(I_2)$.

e. If A, B are invertible $n \times n$-matrices then $A + B$ is invertible. True \ False

counter example: $A = I_3$ and $B = -I_3$. Then A, B are invertible(why?explain!) but $A + B$ is the zero matrix which is not invertible.

Introductory Linear Algebra-Midterm II
Preparation questions

MATH 250 (Instructor: Tom Benhamou) September 19, 2024

f. Let A be an $n \times (n+1)$-matrix such that for every $\bar{b} \in \mathbb{R}^n$, $Ax = \bar{b}$ has a solution, then erasing the last column from A results in an invertible martix. True \ False

counter example: $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

explanation: for any $\bar{b}$, $A \cdot \bar{x} = \bar{b}$ has a solution (since it is already in Eachelon form and there are no 0 rows!) but erasing the last column of A results in $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ which is not invertible (for example- the determinant is 0).

Introductory Linear Algebra-Midterm II
Preparation questions

MATH 250 (Instructor: Tom Benhamou) September 19, 2024

Problem 2. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is given by $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x+2y \\ 2x-y \\ 3x+3y \end{bmatrix}$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ is given by $S(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x-z \\ z-2y-x \end{bmatrix}$. What is the standard matrix of $T, S, T \circ S, S \circ T$? Is $T \circ S$ one-to-one? is it onto?

solution. The standard matrix of T is $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 3 \end{bmatrix}$ and for S it is $B = \begin{bmatrix} 1 & 0 & -1 \\ -1 & -2 & 1 \end{bmatrix}$. Then by a theorem we saw in class the standard matrix for $T \circ S$ is

$$A \cdot B = \begin{bmatrix} -1 & -4 & 1 \\ 3 & 2 & -3 \\ 0 & -6 & 0 \end{bmatrix}$$

and the standard matrix for $S \circ T$ is

$$B \cdot A = \begin{bmatrix} -2 & -1 \\ -2 & 3 \end{bmatrix}$$

To check whether $T \circ S$ is one-to-one, by a theorem we saw in class it suffices to check whether the equation $(AB)\bar{x} = 0$ has a unique solution. Eliminating

$$\begin{bmatrix} -1 & -4 & 1 \\ 3 & 2 & -3 \\ 0 & -6 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & -4 & 1 \\ 0 & -10 & 1 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & -4 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we see that every column has a leading entry and therefore there is a unique solution.

Problem 3. Let A_θ be the standard matrix for the rotation map $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ by θ^o counterclockwise around the origin. What is $\det(A_\theta)$? [hint: you may use trigonometric identities]

solution: As we have seen in class, the standard matrix for T_θ matrix is

$$A_\theta = \begin{bmatrix} \sin(\theta) & -\cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{bmatrix}$$

the determinant of A_θ, by the formula of a 2×2-determinant is

$$\det(A_\theta) = \sin(\theta)^2 + \cos(\theta)^2 = 1$$

the last equality is by the Pythagorean identity from trigonometry.

Problem 4. Show that any $n \times n$-matrix A satisfying $2A^2 + 3A - 5I_3 = 0$ is invertible.

Proof. By properties of matrix algebra, If

$$2A^2 + 3A - 5I_n = 0$$

then

$$2A^2 + 3A = 5I_n$$

therefore

$$(2A + 3I_n)A = 5I_n$$

multiplying both sides by 15 we get

$$\frac{1}{5}(2A + 3I_n)A = I_n$$

So we found a matrix $C = \frac{1}{5}(2A + 3I_n)$ such that $CA = I_n$, namely, A is left invertible. Since A is square, as we have seen in class this is equivalent to A being invertible $\square$

Problem 5. Show that $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -5 & 3 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$ is invertible, compute $\det(A)$ and compute A^{-1}.

Solution. By the algorithm we saw in class, we need to show that A is row equivalent to I and by eliminating we will get $[A|I] \longrightarrow [I|A^{-1}]$

$$\begin{bmatrix} 1 & -1 & 1 & | & 1 & 0 & 0 \\ 0 & -5 & 3 & | & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & | & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 1 & | & 1 & 0 & 0 \\ 0 & -5 & 3 & | & 0 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & | & -\frac{1}{2} & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} & | & \frac{1}{2} & 0 & 1 \\ 0 & 0 & \frac{1}{2} & | & -\frac{5}{2} & 1 & 5 \\ 0 & 1 & -\frac{1}{2} & | & -\frac{1}{2} & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} & | & \frac{1}{2} & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & | & -\frac{1}{2} & 0 & 1 \\ 0 & 0 & \frac{1}{2} & | & -\frac{5}{2} & 1 & 5 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & | & 3 & -1 & -4 \\ 0 & 1 & 0 & | & -3 & 1 & 6 \\ 0 & 0 & \frac{1}{2} & | & -\frac{5}{2} & 1 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & | & 3 & -1 & -4 \\ 0 & 1 & 0 & | & -3 & 1 & 6 \\ 0 & 0 & 1 & | & -5 & 2 & 10 \end{bmatrix}$$

Since A is row redusible to I, A is invertible and by the above computation $A^{-1} = \begin{bmatrix} 3 & -1 & -4 \\ -3 & 1 & 6 \\ -5 & 2 & 10 \end{bmatrix}$ To compute the determinant of A, we note that the first two steps of the above reduction only used operations of adding a multiplicity of a row to another row, which does note change the value of the determinant, and then we swapped rows 2 and 3. Thus,

$$\det(A) = \det(\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}) = \frac{1}{2}$$

The last equality is true since the determinant of a triangular matrix is the product of the values on the diagonal as we have seen in class.

Problem 6. Find all values h for which $A = \begin{bmatrix} h & 1 & 2 \\ 1 & 0 & h \\ -1 & 2 & 1 \end{bmatrix}$ is invertible.

Solution: By a theorem from class, A is invertible if and only if $\det(A) \neq 0$, so we will have to check for which values of h, $\det(A) \neq 0$. We use the determinant expansion using the second column

$$\det(\begin{bmatrix} h & 1 & 2 \\ 1 & 0 & h \\ -1 & 2 & 1 \end{bmatrix}) = -1 \cdot \det(\begin{bmatrix} 1 & h \\ -1 & 1 \end{bmatrix}) - 2 \cdot \det(\begin{bmatrix} h & 2 \\ 1 & h \end{bmatrix}) = -(1+h) - 2(h^2-2) = -2h^2 - h + 3$$

We need to find all h such that $-2h^2 - h + 3 \neq 0$, by the usual root formula we get $h \neq 1, -\frac{3}{2}$.

Problem 7. Solve the following linear system using Cramer's rule.

$$\begin{cases} 3x_1 + x_2 - x_3 = 2 \\ 2x_1 + x_3 = -1 \\ -x_1 + x_2 + x_3 = 1 \end{cases}$$

solution: Write this system as a matrix equation:

$$\begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

By Cramer's rule, the solution is given by $x_i = \frac{\det(A_i(b))}{\det(A)}$ so we need to compute the following determinants:

$$\det(A) = \det(\begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}) = -\det(\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}) - 1\det(\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}) = -1(3) - 1(5) = -8$$

$$\det(A_1(b)) = \det(\begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}) = -\det(\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}) - 1\det(\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}) = -1(-2) - 1(1) = 1$$

$$\det(A_2(b)) = \det(\begin{bmatrix} 3 & 2 & -1 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}) = \det(\begin{bmatrix} 0 & 5 & 2 \\ 0 & 1 & 3 \\ -1 & 1 & 1 \end{bmatrix}) - \det(\begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}) = -(15-2) = -13$$

$$\det(A_3(b)) = \det(\begin{bmatrix} 3 & 1 & 2 \\ 2 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}) = -\det(\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}) - \det(\begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix}) = -(2-1) - (-7) = 6$$

Hence the unique solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{8} \\ \frac{13}{8} \\ -\frac{3}{4} \end{bmatrix}$

Introductory Linear Algebra-Midterm II Preparation questions

MATH 250 (Instructor: Tom Benhamou) September 19, 2024

Problem 8. (a) Express the solution set of the following homogeneous system as the span of vectors:

$$\begin{cases} 3x_1 - x_3 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 - 2x_2 - 3x_3 - 2x_4 = 0 \end{cases}$$

(b) Find the general solution to the non-homogeneous equation:

$$\begin{cases} 3x_1 - x_3 = 1 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 - 2x_2 - 3x_3 - 2x_4 = 1 \end{cases}$$

express your solution using a private solution and the general solution to the homogeneous system.

Solution. (a) To find the solutions to the homogeneous equation we reduce the coefficient matrix

$$\begin{bmatrix} 3 & 0 & -1 & 0 \\ 2 & 1 & 1 & 1 \\ -1 & -2 & -3 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & -6 & -10 & -6 \\ 0 & -3 & -5 & -3 \\ -1 & -2 & -3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -3 & -5 & -3 \\ 0 & -6 & -10 & -6 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -3 & -5 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 5/3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/3 & 0 \\ 0 & 1 & 5/3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So x_3, x_4 are free variables and $x_1 = \frac{x_3}{3}$, $x_2 = -\frac{5}{3}x_3 - x_4$. So the general solution is:

$$\begin{bmatrix} \frac{x_3}{3} \\ -\frac{5}{3}x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} \frac{1}{3} \\ -\frac{5}{3} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Namely the set of solutions to the homogeneous equation is

$$\text{Span}(\begin{bmatrix} \frac{1}{3} \\ -\frac{5}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix})$$

(b) By the theorem we saw in class, the general solution to the non-homogeneous equation is just a private solution for the non-homogeneous+ the general solution to the homogeneous equation. Let us find a private solution:

$$\begin{bmatrix} 3 & 0 & -1 & 0 & |1 \\ 2 & 1 & 1 & 1 & |0 \\ -1 & -2 & -3 & -2 & |1 \end{bmatrix} \to \begin{bmatrix} 0 & -6 & -10 & -6 & |4 \\ 0 & -3 & -5 & -3 & |2 \\ -1 & -2 & -3 & -2 & |1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 2 & |-1 \\ 0 & -3 & -5 & -3 & |2 \\ 0 & -6 & -10 & -6 & |4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 3 & 2 & |-1 \\ 0 & -3 & -5 & -3 & |2 \\ 0 & 0 & 0 & 0 & |0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 2 & |-1 \\ 0 & 1 & 5/3 & 1 & |-\frac{2}{3} \\ 0 & 0 & 0 & 0 & |0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/3 & 0 & |\frac{1}{3} \\ 0 & 1 & 5/3 & 1 & |-\frac{2}{3} \\ 0 & 0 & 0 & 0 & |0 \end{bmatrix}$$

So $\begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 0 \\ 0 \end{bmatrix}$ is a private solution and therefore the general solution of the non-homogeneous system is

$$\begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{1}{3} \\ -\frac{5}{3} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Problem 9. Show that a square matrix A is invertible if and only if A^2 is invertible.

Proof. By an theorem from class, A is invertible if and only if $\det(A) \neq 0$ which is if and only if $\det(A)^2 \neq 0$, but since $\det(A^2) = \det(A)^2$ (since the determinant is multiplicative), this is if and only if A^2 is invertible. □

Problem 10. Show that for any square matrix A, $\det(A \cdot A^T) \geq 0$.

Proof. We have $\det(A \cdot A^T) = \det(A) \cdot \det(A^T)$ since the determinant is multiplicative. Also we have seen in class that $\det(A^T) = \det(A)$ hence $\det(A \cdot A^T) = \det(A)^2$, and since a sequence of a real number is always non negative we get $\det(A \cdot A^T) \geq 0$. □