Introductory Linear Algebra-Midterm II

MATH 250 (Instructor: Tom Benhamou) October 28, 2024

Instruction

The midterm consists of 3 problems, each worth 34 points (The maximal grade is 100). For this you will have one hour. No material is allowed. The solutions to the problems should be written in the designated areas and the "extra page" at the end. Detailed explanations for your solutions are required unless stated otherwise.

Full Name (PRINT): ______________________________

Net ID: ______________________________

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Problems

Problem 1. For each of the following statements determine if it is true are false. Provide a counterexample if false. No explanation is required if true (circle the correct answer):

a. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, then $\{T(\bar{e}_1), T(\bar{e}_2)\}$ are linearly independent. True \ False

   counter example: $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(\bar{x}) = \bar{0}$.

b. If A is an invertible martrix and $\alpha \neq 0$ is a scalar, then $\alpha \cdot A$ is invertible. True \ False

   Explanation: Either computing the determinant, or noticing that $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$.

c. If A, B are non square matrices then $A \cdot B$ is not invertable. True \ False

   counter example: For example $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

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Problem 2. Suppose that $T:\mathbb{R}^2\to\mathbb{R}^3$ is given by $T(\begin{bmatrix}x\\y\end{bmatrix})=\begin{bmatrix}-2y\\3x+y\\x+3y\end{bmatrix}$ and $S:\mathbb{R}^3\to\mathbb{R}^2$ is given by $S(\begin{bmatrix}x\\y\\z\end{bmatrix})=\begin{bmatrix}2x-2z\\z+2x+y\end{bmatrix}$.

a. Find the standard matrix A of the linear transformation $T\circ S$.

b. Compute $\det(A)$.

c. Is $T\circ S$ invertible? circle your answer – no explanation is required

YES \ <u>NO</u>

Solution: a. By definition, the standard matrix for T is

$$A_0=\begin{bmatrix}T(\begin{bmatrix}1\\0\end{bmatrix}) & T(\begin{bmatrix}0\\1\end{bmatrix})\end{bmatrix}=\begin{bmatrix}0&-2\\3&1\\1&3\end{bmatrix}$$

Similarily, the standard matrix for S is

$$A_1=\begin{bmatrix}S(\begin{bmatrix}1\\0\\0\end{bmatrix}) & S(\begin{bmatrix}0\\1\\0\end{bmatrix}) & S(\begin{bmatrix}0\\0\\1\end{bmatrix})\end{bmatrix}=\begin{bmatrix}2&0&-2\\2&1&1\end{bmatrix}$$

From a theorem we saw in class, the standard matrix for the composition $T\circ S$ is given by $A_0\cdot A_1$ which is

$$\begin{bmatrix}0&-2\\3&1\\1&3\end{bmatrix}\cdot\begin{bmatrix}2&0&-2\\2&1&1\end{bmatrix}=\begin{bmatrix}-4&-2&-2\\8&1&-5\\5&3&1\end{bmatrix}$$

Solution: b. We claim that $\begin{bmatrix} -4 & -2 & -2 \\ 8 & 1 & -5 \\ 8 & 3 & 1 \end{bmatrix}$ is not invertible and therefore $\det(A) = 0$. To see this, we eliminate

$$\begin{bmatrix} -4 & -2 & -2 \\ 8 & 1 & -5 \\ 8 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} -4 & -2 & -2 \\ 0 & -3 & -9 \\ 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} -4 & -2 & -2 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} -4 & -2 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there is a 0-line, the matrix A does not reduce to I and therefore not invertible. Hence $\det(A) = 0$.

c. Since A is the standard matric of $T \circ S$ and A is not invertible it follows that $T \circ S$ is not invertible.

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Problem 3. Find all the values h for which $A = \begin{bmatrix} 2 & h & 3 \\ 1 & 0 & h \\ -1 & 2 & 1 \end{bmatrix}$ is invertible. Then for each such value h, compute A^{-1}.

solution: The matrix A is invertible if and only if $\det(A) \neq 0$. Computing $\det(A)$, we use the second column expansion:

$$\det(\begin{bmatrix} 2 & h & 3 \\ 1 & 0 & h \\ -1 & 2 & 1 \end{bmatrix}) = -h\det(\begin{bmatrix} 1 & h \\ -1 & 1 \end{bmatrix}) - 2\det(\begin{bmatrix} 2 & 3 \\ 1 & h \end{bmatrix}) =$$

$$= -h(1+h) - 2(2h-3) = -h^2 - 5h + 6 = (h-1)(-6-h)$$

Hence A if invertible if and only if $\det(A) \neq 0$ if and only if $h \neq 1, -6$.

Given any $h \neq 1, -6$ we compute A^{-1} using Cramer's rule:

$$A^{-1} = \frac{1}{\det(A)} Adj(A)$$

so let us compute $B = Adj(A)$

$B_{11} = \det(\begin{bmatrix} 0 & h \\ 2 & 1 \end{bmatrix}) = -2h$, $B_{12} = -\det(\begin{bmatrix} h & 3 \\ 2 & 1 \end{bmatrix}) = 6 - h$

$B_{13} = \det(\begin{bmatrix} h & 3 \\ 0 & h \end{bmatrix}) = h^2$, $B_{21} = -\det(\begin{bmatrix} 1 & h \\ -1 & 1 \end{bmatrix}) = -1 - h$

$B_{22} = \det(\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}) = 5$ $B_{23} = -\det(\begin{bmatrix} 2 & 3 \\ 1 & h \end{bmatrix}) = -2h + 3$

$B_{31} = \det(\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}) = 2$, $B_{32} = det(\begin{bmatrix} 2 & h \\ -1 & 2 \end{bmatrix}) = -4 - h$

$B_{33} = \det(\begin{bmatrix} 2 & h \\ 1 & 0 \end{bmatrix}) = -h$

Hence $A^{-1} = \frac{1}{-h^2-5h+6}\begin{bmatrix} -2h & 6-h & h^2 \\ -1-h & 5 & -2h+3 \\ 2 & -4-h & -h \end{bmatrix}$.

Extra Page: