ISOMORPHISM CLASSES OF GENERATING SETS

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Abstract. We prove that for any two regular cardinals $\omega < \lambda_0 < \lambda_1$ there is a ccc forcing extension where there is an ultrafilter U on ω with a base $\mathcal{B}$ such that $(\mathcal{B}, \supseteq^*) \cong \lambda_0 \times \lambda_1$. We use similar ideas to construct an ultrafilter with a base $\mathcal{B}$ as above which is order isomorphic to any given two-dimensional, well-founded, countably directed order with no maximal element. Similarly, relative to a supercompact cardinal, it is consistent that κ is supercompact, and for any regular cardinals $\kappa < \lambda_0 < \lambda_1 < ... < \lambda_n$, there is a $<\kappa$-directed closed κ^+-cc forcing extension where there is a normal ultrafilter U on κ with a base $\mathcal{B}$ such that $(\mathcal{B}, \supseteq^*) \cong \lambda_0 \times ... \times \lambda_n$. We apply our constructions to obtain ultrafilters with controlled Tukey-type, in particular, an ultrafilter with non-convex Tukey and depth spectra is presented, answering questions from [4]. Our construction also provides new models where $\mathfrak{u}_\kappa < 2^\kappa$.

1. Introduction

Ultrafilters with strong combinatorial properties have been proven useful for the construction of mathematical objects with extreme behavior such as ultraproducts, topological spaces (and special points in topological spaces), and certain groups, but could also be used to prove some combinatorial properties [32, 28, 34]. Many of these combinatorial properties are obtained by constructing a special generating set for the ultrafilter. This paper aims to study the possible structures of such generating sets. More precisely, we are interested in the isomorphism class of the structure $(\mathcal{B}, \supseteq^*)$, where $\supseteq^*$ is the reversed inclusion modulo the bounded ideal (or finite in case of ω) and $\mathcal{B}$ is a $\subseteq^*$-generating set of an ultrafilter U, namely, $\mathcal{B} \subseteq U$ and for every $A \in U$, there is $B \in \mathcal{B}$ such that $B \subseteq^* A$.

For example, Kunen [33] used an iteration of Mathias forcing relative to an ultrafilter from [35] to construct ultrafilters on ω with generating sets which are $\subseteq^*$-decreasing of any desired ordinal length of uncountable cofinality λ, these are called simple P_λ-points. Kunen used this to separate the ultrafilter

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number $\mathfrak{u}$ from the continuum, where $\mathfrak{u}$ is defined to be the least cardinality of a generating set for a uniform ultrafilter on ω. Our main construction generalizes Kunen's construction to form generating sets of more complex types. In particular, we prove the following consistency result:

Theorem 1.1. *Assume GCH and let λ_0, λ_1 be cardinals of uncountable cofinality. Then there is a ccc forcing extension in which there is an ultrafilter U on ω, with a $\subseteq^*$-generating set $\mathcal{B} \subseteq U$ such that $(\mathcal{B}, \supseteq^*) \simeq \lambda_0 \times \lambda_1$.*

The proof uses non-linear iterations of Mathias forcing. Such iterations were considered before by Hechler [29] who proved a similar result to obtain dominating families in the structure $(\omega^\omega, \leq^*)$. Several results in the spirit of Hechler can be found in the literature [23, 17, 1]. Blass and Shelah [13] considered a non-linear iteration involving Mathias forcing, however, the Mathias component only really appears in one dimension of this construction. The major difficulty, however, is to obtain at the end of the process (i.e. iteration) a uniform ultrafilter.

In an attempt to realize other orders, and generalizing Kunen's simple P_λ-point, we define:

Definition 1.2. Given an ordered set $\mathbb{D} = (D, \leq_D)$, we say that a filter F is a simple $P_{\mathbb{D}}$-point if there is a $\subseteq^*$-generating set $\mathcal{B} \subseteq F$ such that $(\mathcal{B}, \supseteq^*) \simeq \mathbb{D}$.

Hence Theorem 1.1 says that in ccc forcing extension, there is a uniform simple $\mathbb{P}_{\lambda_0 \times \lambda_1}$-point ultrafilter. Our construction applies to so-called *degree 2*-lattices, from which we obtain a more general construction for two-dimensional orders. The dimension of a partially ordered set $\mathbb{D} = (D, \leq_D)$ was defined by Dushnik and Miller [21] as the least n such that $\mathbb{D}$ can be embedded into the Cartesian product of n-many linear orders with the pointwise ordering (For a more detailed account see [45]).

Theorem 1.3. *Suppose that $\mathbb{D}$ is a well-founded, two-dimensional, countably directed lattice with no maximal element, then there is a ccc extension in which there is a simple $P_{\mathbb{D}}$-point ultrafilter on ω.*

Generalizing Kunen's result to measurable cardinals requires considerable effort and does not run smoothly using the standard indestructibility methods. In an unpublished work, Carlson showed that from a supercompact cardinal this is possible. Recently, this type of iteration appeared in several constructions [27, 18, 16, 22] guided by (for example) a diamond sequence so that a supercompact embedding lifts in a way that the normal ultrafilter derived from the lifted embedding has a $\subseteq^*$-decreasing generating set of the desired length. This can be used to obtain a model with $2^\kappa > \kappa^+$, with a κ-complete ultrafilter over κ generated by fewer than 2^κ-many sets. It remains open whether this can be achieved from weaker assumptions.

Our results also apply to this type of lifting argument. In particular, we obtain the following:

Theorem 1.4. *Relative to the existence of a supercompact cardinal, there is a model V with a supercompact cardinal κ such that for every regular cardinals $\kappa < \lambda_0 < \lambda_1$, there is a $<\kappa$-directed closed κ^+-cc forcing extension in which there is a κ-complete simple $P_{\lambda_0\times\lambda_1}$-point ultrafilter on κ.*

Again, we can use general order theoretic reductions to obtain further dimension-two orders as simple P-point ultrafilters. However, while the construction on ω does not work for three-dimensional orders (such as $\lambda_0 \times \lambda_1 \times \lambda_2$), the diamond mechanism is more flexible in that sense and in fact we obtain the following generalization:

Theorem 1.5. *Relative to the existence of a supercompact cardinal, there is a model V with a supercompact cardinal κ such that for all regular cardinals $\kappa < \lambda_0 < \lambda_1 < ... < \lambda_n$, there is a $<\kappa$-directed closed κ^+-cc forcing extension in which there is a κ-complete simple $P_{\lambda_0\times\lambda_1\times...\times\lambda_n}$-point ultrafilter over κ.*

These results can also be used to construct complex generating sets for the club filter on a measurable cardinal:

Corollary 1.6. *Relative to the existence of a supercompact cardinal, there is a model V with a supercompact cardinal κ such that for every regular cardinals $\kappa < \lambda_0 < \lambda_1 < ... < \lambda_n$, there is a $<\kappa$-directed closed κ^+-cc forcing extension in which κ is measurable and the club filter is a simple $P_{\lambda_0\times...\times\lambda_n}$-point.*

Finally, we apply these constructions to control the Tukey-type of certain ultrafilters. Recall that the Tukey order on directed sets is defined by $(\mathbb{P}, \leq_P) \leq_T (\mathbb{Q}, \leq_Q)$ if there is a Tukey reduction $f : \mathbb{P} \to \mathbb{Q}$, that is, for every unbounded $\mathcal{B} \subseteq \mathbb{P}$, $f[\mathcal{B}]$ is unbounded in $\mathbb{Q}$. We say that $\mathbb{P} \equiv_T \mathbb{Q}$ if $\mathbb{P} \leq_T \mathbb{Q}$ and $\mathbb{Q} \leq_T \mathbb{P}$. In this paper, we will only consider ultrafilters U ordered by reversed inclusion $(U, \supseteq)$ and almost inclusion $(U, \supseteq^*)$. The study of the Tukey order find its origins in the concept of Moore-Smith convergence of nets and has been studied extensively on general ordered sets [46, 30, 44]. The Tukey-type $(U, \supseteq)$ also has been subject to a considerable amount of work, especially when U is an ultrafilter on ω [30, 19, 20, 12, 38, 39, 36, 11, 3] and lately this has been also looked at in the context of measurable cardinals [5, 9] due to its close relation to the Galvin property and applicability to Prikry-type forcing theory [8, 26, 6]. The Tukey-type of $(U, \supseteq^*)$ as also been of interest [36, 5] as the properties of a $\subseteq^*$-generating set for an ultrafilter are related to this Tukey-type. In fact, most of the combinatorial characteristics of ultrafilters (see for example [15]) are formulated in terms of $\subseteq^*$ rather than $\subseteq$. The connection to the Tukey order is noticeable once the Tukey order is expressed in-terms of cofinal maps. Given a partially ordered set $(D, \leq_D)$, a set $B \subseteq D$ is called cofinal in $(D, \leq_D)$ if for every $d \in D$ there is $b \in B$ such that $d \leq_D b$. Schmidt Duality [40] in the special case of ultrafilters (or in the situation where there are least upper bounds): For any two ultrafilters U, W, $U \leq_T W$ if and only if there is a function $f : W \to U$ which is monotone (i.e. $A \subseteq B \Rightarrow f(A) \subseteq f(B)$) and has a cofinal image in U.

It is not hard to check that whenever U is a simple $P_{\mathbb{D}}$-point, then $(U, \supseteq^*) \equiv_T \mathbb{D}$. So, the analysis in this paper is finer than the Tukey-type analysis of ultrafilters, but can be used to control it. For example, we use our construction to find an ultrafilter U which is Tukey-equivalent to $\lambda_0 \times \lambda_1$. From this we can analyze the Tukey spectrum and the depth spectrum (see Definitions 5.1, 5.6) and show the consistency of an ultrafilter with a non-convex spectrum (both depth and Tukey). This was asked by the first author in [4].

Theorem 1.7.

(1) *It is consistent that there is an ultrafilter U on ω such that* $\mathrm{Sp}_T(U)$ *and* $\mathrm{Sp}_{dp}(U)$ *are non-convex sets.*
(2) *Relative to the existence of a supercompact cardinal, it is consistent that there is a normal ultrafilter U such that* $\mathrm{Sp}_T(U)$ *and* $\mathrm{Sp}_{dp}(U)$ *are non-convex sets.*

Our second application relates to generalized cardinal characteristics. Recall that

$$\mathfrak{b}_\kappa = \min\{|\mathcal{F}| \mid \mathcal{F} \subseteq \kappa^\kappa \text{ is } \leq^*\text{-unbounded}\}$$
$$\mathfrak{d}_\kappa = \min\{|\mathcal{F}| \mid \mathcal{F} \subseteq \kappa^\kappa \text{ is } \leq^*\text{-dominating}\}$$
$$\mathfrak{u}_\kappa = \min\{|\mathcal{B}| \mid \mathcal{B} \subseteq^*\text{-generates a uniform ultrafilter on } \kappa\}$$
$$\mathfrak{u}_\kappa^{com} = \min\{|\mathcal{B}| \mid \mathcal{B} \subseteq^*\text{-generates a } \kappa\text{-complete ultrafilter on } \kappa\}$$

As we mentioned before, at a measurable cardinal, the only known method to produce models where $\mathfrak{u}_\kappa < 2^\kappa$ is via the linear Mathias iteration. The computation of the values of these cardinal characteristics was done in [16] where it was shown that after a linear iteration of Mathias forcing, $\mathfrak{u}_\kappa = \mathfrak{u}_\kappa^{com} = \mathfrak{b}_\kappa = \mathfrak{d}_\kappa$. Lately, the first and the third author [10] generalized this and showed that merely the presence of a simple P_λ-point (which is the kind of ultrafilters produced by the Mathias iteration) causes $\mathfrak{u}_\kappa = \mathfrak{u}_\kappa^{com} = \mathfrak{b}_\kappa = \mathfrak{d}_\kappa$. So the question of separating the cardinal characteristics in a model where $\mathfrak{u}_\kappa < 2^\kappa$ remains open. Using our methods, we can prove the following:

Theorem 1.8. *Relative to the existence of a supercompact cardinal, it is consistent that κ is measurable and $\mathfrak{b}_\kappa < \mathfrak{d}_\kappa = \mathfrak{u}_\kappa < 2^\kappa$. In particular, there is no simple P-point in that model.*

2. Some preliminaries

Given a partial order $(D, \leq_D)$, we define $d_1 <_D d_2$ if $d_1 \leq_D d_2$ and $d_1 \neq d_2$. For each $d \in D$, let $D_{<d} = \{e \in D : e <_D d\}$; $D_{\leq d}$ is defined similarly.

In this paper, we would like to study what are the possible posets $\mathbb{D}$ for which there are uniform simple $P_{\mathbb{D}}$-point ultrafilters, both on ω and on measurable cardinals. For the following easy proposition, we define that a poset $\mathbb{P} = (P, \leq_P)$ embeds cofinally into a poset $\mathbb{Q} = (Q, \leq_Q)$ if there is an order preserving injection $j : \mathbb{P} \to \mathbb{Q}$ such that $\mathrm{rng}(j)$ is a cofinal subset of $\mathbb{Q}$.

Proposition 2.1. *Let U be an ultrafilter. If U is a simple $P_{\mathbb{D}}$-point and $\mathbb{D}'$ is embedded cofinally in $\mathbb{D}$, then U is a simple $P_{\mathbb{D}'}$-point.*

There are some trivial limitations on a poset $\mathbb{D}$ for which there can be a simple $P_{\mathbb{D}}$-point ultrafilter. For example, $\mathbb{D}$ has to be directed. Also, $\mathbb{D}$ cannot have maximal elements. The following limitation generalizes [15, Prop. 1.4].

Theorem 2.2. *Let U be a uniform κ-complete ultrafilter over $\kappa \geq \omega$. If U is a simple $P_{\mathbb{D}}$-point, then $\mathbb{D}$ is not the increasing union of κ-many κ-directed non-cofinal subsets of $\mathbb{D}$.*

Proof. Suppose otherwise, that $\mathbb{D} = (D, \leq_D)$ and let $D = \bigcup_{i<\kappa} D_i$, where $\langle D_i \mid i < \kappa\rangle$ is an $\subseteq$-increasing sequence of κ-directed non-cofinal subsets of D. By moving to a subsequence of the D_i's if necessary, we may assume that for every i, there is an element e_i of D_{i+1} which is not bounded by any element of D_i.

Let $\mathcal{B} = (b_d)_{d\in D}$ be a $\subseteq^*$-generating set for U witnessing that U is a simple $P_{\mathbb{D}}$-point. Let F_i be the κ-complete filter generated by $(b_d)_{d\in D_i}$. Note that since D_i is κ-directed, $(b_d)_{d\in D_i}$ is a $\subseteq^*$-generating set for F_i. Let $F = \bigcup_{i<\kappa} F_i$ and let us argue that $F = U$. Indeed, the filters F_i are increasing (as the D_i's are increasing) and therefore F is a filter. Clearly, $\mathcal{B}$, which is a generating set for U, is included in F and by maximality of ultrafilters, $F = U$.

Recall that $e_i \in D_{i+1}$ is not bounded by any element of D_i and consider $A_i = \bigcap_{j\leq i} b_{e_j}$. Then:

Claim 2.3.

(1) *The sequence $\langle A_i \mid i < \kappa\rangle$ is $\subseteq$-decreasing.*
(2) *$A_i \in F_r$ for every $r > i$.*
(3) *$A_i \notin F_i$*
(4) *$\bigcap_{i<\kappa} A_i \notin U$*

Proof of claim. (1) is trivial. For (2), by the choice of e_j, $e_j \in D_{j+1}$. Since the D_r's are increasing, $e_j \in D_r$ for every $r > j$. Hence for every $j \leq i$ and every $r > i$, $b_{e_j} \in F_r$. By κ-completeness, $A_i = \bigcap_{j\leq i} b_{e_j} \in F_r$. For (3), suppose otherwise that $A_i \in F_i$. Since $(b_d)_{d\in D_i}$ generates F_i, we can find $d \in D_i$ such that $b_d \subseteq^* A_i$. But then $b_d \subseteq^* b_{e_i}$, and since $((b_d)_{d\in D}, \supseteq^*) \simeq \mathbb{D}$ we conclude that $e_i \leq_D d \in D_i$. This contradicts the choice of e_i.

(4) follows easily from (3) and the fact that $\bigcup_{i<\kappa} F_i = U$. $\square_{\text{claim}}$

Consider the set $A^* = A_0 \setminus \bigcap_{i<\kappa} A_i$. By (4) of the claim, $A^* \in U$. Let $C_\alpha = (\bigcap_{\beta<\alpha} A_\beta) \setminus A_\alpha$. Using (1), it is not hard to check that A^* can be partitioned as $A^* = \biguplus_{1\leq\alpha<\kappa} C_\alpha$.

Next, we split. Define A^*:

$$X_{\text{even}} = \bigcup_{\alpha<\kappa} C_{2\alpha},\ X_{\text{odd}} = \bigcup_{\alpha<\kappa} C_{2\alpha+1}$$

Since $X_{\text{even}} \cup X_{\text{odd}} = A^* \in U$, either $X_{\text{even}} \in U$ or $X_{\text{odd}} \in U$. Suppose for example that $X_{\text{even}} \in U$ (the argument in the case that $X_{\text{odd}} \in U$ is identical). Then there is $i_1 < \kappa$ such that $X_{\text{even}} \in F_{i_1}$. By (2) of the claim, and κ-completeness of F_{i_1}, $\bigcap_{\alpha<i_1} A_\alpha \in F_{i_1}$. Therefore $X_{\text{even}} \cap \bigcap_{\alpha<i_1} A_\alpha \in F_{i_1}$. Since the F_r's are increasing, we may assume that $i_1 = 2\gamma+1$ for some $\gamma < \kappa$. Finally, we claim that $X_{\text{even}} \cap \bigcap_{\alpha<2\gamma+1} A_\alpha \subseteq A_{2\gamma+1}$, which produces the desired contradiction. Indeed if $\nu \in X_{\text{even}} \cap \bigcap_{\alpha<2\gamma+1} A_\alpha$ then by definition of X_{even} there is $\alpha^* < \kappa$ such that $\nu \in C_{2\alpha^*}$. Since $\nu \in \bigcap_{\alpha<2\gamma+1} A_\alpha$, it follows that $2\alpha^* > 2\gamma+1$. This means that $\nu \in A_{2\gamma+1}$. □

Corollary 2.4. *There is no uniform simple $P_{\omega\times\omega_1}$-point ultrafilter on ω.*

The ultrafilters we will be constructing are all p-points. This poses more restrictions on the possible $\mathbb{D}$ which we will be able to realize. Recall that an ultrafilter U is a P_λ-point, if $(U, \supseteq^*)$ is a λ-directed ordered set.

Proposition 2.5. *U is a P_λ-point iff for every (any) $\mathbb{D}$ such that U is a simple $P_{\mathbb{D}}$-point, $\mathbb{D}$ is λ-directed.*

We now turn to the order theoretic part of the preliminaries. The following definition will be essential in our construction of a simple $P_{\mathbb{D}}$-point with a prescribed $\mathbb{D}$.

Definition 2.6. A partially ordered set $\mathbb{D} = (D, \leq_D)$ is called a *lattice*, if for any $x, y \in D$ there is a least upper bound $x \vee y \in D$ *the join* and a greatest lower bound $x \wedge y \in D$ called *the meet*. A *semi-lattice* admits only meets. We say that a semi-lattice D has *degree* 2 if for any $d \in D$ and any $x_1, x_2, x_3 <_D d$, there are $1 \leq i \neq j \leq 3$ and $e <_D d$ such that $x_i, x_j \leq_D e$.

Given a lattice $\mathbb{D}$, a sublattice $\mathbb{A}$ of $\mathbb{D}$ is a suborder of $\mathbb{D}$ which is closed under joins and meets.

Example 2.7. Any product of ordinals $\lambda_0 \times \lambda_1$ is a lattice of degree 2: Given pairs $(\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3) < (\alpha, \beta)$,

$$\alpha_{\max} = \max_{1\leq i\leq 3} \alpha_i,\ \alpha_{\min} = \min_{1\leq i\leq 3} \alpha_i,\ \beta_{\max} = \max_{1\leq i\leq 3} \beta_i,\ \beta_{\min} = \min_{1\leq i\leq 3} \beta_i.$$

Then

$$(\alpha_1, \beta_1) \vee (\alpha_2, \beta_2) \vee (\alpha_3, \beta_3) = (\alpha_{\max}, \beta_{\max})$$
$$(\alpha_1, \beta_1) \wedge (\alpha_2, \beta_2) \wedge (\alpha_3, \beta_3) = (\alpha_{\min}, \beta_{\min}).$$

If the pairs are pairwise incomparable, then for each $i = 1, 2, 3$, either $\alpha_i < \alpha_{\max}$ or $\beta_i < \beta_{\max}$. By the pigeonhole principle, there are $i \neq j$ such that either $\max\{\alpha_i, \alpha_j\} < \alpha_{\max}$ or $\max\{\beta_i, \beta_j\} < \beta_{\max}$. Without loss of generality suppose that $\max\{\alpha_i, \alpha_j\} < \alpha_{\max}$. This means that

$$(\max(\alpha_i, \alpha_j), \max(\beta_i, \beta_j)) < (\alpha_{\max}, \beta_{\max}) \leq (\alpha, \beta).$$

Hence $\lambda_0 \times \lambda_1$ has degree 2.

Proposition 2.8. (1) *Suppose that $\mathbb{D}$ is a degree 2 lattice and that $\mathbb{A} \subseteq \mathbb{D}$ is a sub-lattice. Then $\mathbb{A}$ is a degree 2 lattice.*
(2) *If $\mathbb{D}'$ is a directed set which embeds into a degree 2 lattice $\mathbb{D}$, then it also embeds cofinally into a degree 2 sub-lattice.*

Proof. To see (1), suppose that $\mathbb{A} = (A, \leq_D)$ is a sub-lattice of $\mathbb{D} = (D, \leq_D)$ and let $a_1, a_2, a_3 <_A a \in A$. Since $\mathbb{D}$ has degree 2, there are $1 \leq i \neq j \leq 3$ and $b < a$ such that $a_i, a_j \leq b$. In particular, $a_i \vee a_j \leq b < a$. Since A is closed under join, $a_i \vee a_j \in A$. Hence A has degree 2.

For (2), let $g : \mathbb{D}' \to \mathbb{D}$ be an embedding. Let A be the closure of $\mathrm{rng}(g)$ under join and meets. That is, A is the increasing union of sets A_n, such that $A_0 = \mathrm{rng}(g)$, and $A_{n+1} = \{a \vee b \mid a, b \in A_n\} \cup \{a \wedge b \mid a, b \in A\}$. Clearly, A is closed under meets and joins. We claim that $\mathrm{rng}(g)$ is cofinal in A. For this, it suffices to prove that for every n, A_n is cofinal in A_{n+1}. Indeed, since D' is directed, also $\mathrm{rng}(g)$ is directed and therefore A_0 is cofinal in A_1, and therefore A_1 stays directed. Inductively assume that $\mathrm{rng}(g)$ is cofinal in A_n, then by directness, it is cofinal in A_{n+1}.

By (1), A is a degree 2 lattice. Hence $\mathbb{D}'$ can be embedded cofinally into a degree 2 lattice. □

Example 2.9. Consider $\mathbb{D} = (\bigcup_{\alpha<\lambda_0}\{\alpha\} \times \omega_\alpha, \leq_D)$, where $\leq_D$, as in the previous example is the pointwise order on pairs of ordinals. Note that for every $(\alpha, \beta), (\gamma, \delta) \in D$, $(\max\{\alpha, \gamma\}, \max\{\beta, \delta\}), (\min\{\alpha, \gamma\}, \min\{\beta, \delta\}) \in D$. Hence it is evident (and follows Proposition 2.8(2)) that the degree 2 property of $\lambda_0 \times (\sup_{\alpha<\lambda_0} \omega_\alpha)$ is inherited to $\mathbb{D}$.

Corollary 2.10. *Suppose that $\mathbb{D}$ is a two-dimensional, directed, well-founded poset, then $\mathbb{D}$ embeds cofinally into a degree 2 sub-lattice of a product of cardinals.*

Proof. We claim that there is an embedding $g : D \to \lambda_0 \times \lambda_1$ for some cardinals $\lambda_0 \times \lambda_1$ and the rest follows from Proposition 2.8(2). First, use the assumption that $\mathbb{D} = (D, \leq_D)$ is two-dimensional, to identify D with a subset of a product of linear orders $(A, \leq_A) \times (B, \leq_B)$. Now since D is well-founded, we can define g by well-founded recursion, for $(a, b) \in D$, set $g(a, b) = (\gamma, \delta)$, where

$$\gamma = \sup\{\alpha + 1 \mid \exists (a', b') \in D_{<(a,b)},\ a' <_A a, \exists\beta, g(a', b') = (\alpha, \beta)\},$$
$$\delta = \sup\{\beta + 1 \mid \exists (a', b') \in D_{<(a,b)},\ b' <_B b, \exists\alpha, g(a', b') = (\alpha, \beta)\}.$$

□

3. Finite support D-based iteration

In this section we will describe a forcing iteration which is indexed by a non-linear partial order. As we denoted in the introduction, similar iterations appeared in various places in the literature. Notably, in the works of Shelah (see, for example, [41, 42]), this type of iteration is modeled as linear iteration with restricted memory — so in Shelah's language, asking the memory to be transitive is parallel to forcing along a partial order. In [42], Shelah discribed a more general mechanism that allows one to deal with certain schemes which might be ill–founded or non-transitive.

Let $\mathbb{D} = (D, \leq_D)$ be a partial order, a sequence $\langle \mathbb{P}_{<d}, \dot{\mathbb{Q}}_d : d \in D\rangle$ is a *finite support iteration based on* $\mathbb{D}$ if the following conditions hold for all $d \in D$:

- $\mathbb{P}_{<d}$ is the set of finitely supported sequences $p = \langle p_e \mid e \in D_{<d}\rangle$ such that for all $e \in D_{<d}$, $\Vdash_{\mathbb{P}_{<e}} p(e) \in \dot{\mathbb{Q}}_e$.
- $1_{\mathbb{P}_{<d}} \Vdash \dot{\mathbb{Q}}_d$ is a partial order.

We denote by $\mathbb{P}_{\mathbb{D}}$ the set of finitely supported sequences $p = \langle p_e \mid e \in D\rangle$ such that for all $e \in D$, $p \restriction D_{<e} \Vdash_{\mathbb{P}_{<e}} p(e) \in \dot{\mathbb{Q}}_e$. We will also write $\mathbb{P}_{\leq d}$ instead of $\mathbb{P}_{<d} * \dot{\mathbb{Q}}_d$. The order is defined by $p \leq_{\mathbb{P}_{\mathbb{D}}} q$ iff for every $d \in D$, $p \restriction D_{<d} \Vdash_{\mathbb{P}_{<d}} p(d) \leq_{\dot{\mathbb{Q}}_d} q(d)$.

Ultimately, we will only consider iterations based on well-founded partially ordered sets, as the names $\dot{\mathbb{Q}}_d$ will be defined by well-founded recursion.

Lemma 3.1. *Let $\mathbb{D}' \subseteq \mathbb{D}$ be downwards closed. Then $P_{\mathbb{D}}$ projects via restriction to $\mathbb{P}_{\mathbb{D}'}$.*

Proof. We will check that the restriction map from $\mathbb{P}_{\mathbb{D}}$ to $\mathbb{P}_{\mathbb{D}'}$ is a projection. Clearly, it is order preserving. Suppose that $q \in \mathbb{P}_{D'}$ is stronger than $p \restriction D'$. We will find an extension $p' \leq p$ in $\mathbb{P}_{\mathbb{D}}$ such that $p' \restriction D' = q$.

Define $p' = q \cup p \restriction (D \setminus D')$. Then by definition, p' belongs to $\mathbb{P}_{\mathbb{D}}$ Also $p' \restriction D' = q$. It remains to prove that for every $d \in D$,

$$p' \restriction D_{<d} \Vdash_{\mathbb{P}_{<d}} p'(d) \leq p(d).$$

If $d \in D'$, since D' is downward closed, $p' \restriction D_{<d} = q \restriction D_{<d}$ and $p'(d) = q(d)$. Since q is stronger than $p \restriction D'$, we are done.

Suppose instead that $d \in D \setminus D'$. Then $p'(d) = p(d)$, and clearly, $p' \restriction D_{<d} \Vdash_{\mathbb{P}_{<d}} p'(d) \leq p(d)$. □

Recall that the *Mathias forcing relative to a filter* is defined as follows:

Definition 3.2. Let F be a κ-complete filter over a regular cardinal $\kappa \geq \omega$. Let $\mathbb{M}(F)$ be the forcing notion consisting of conditions of the form $(a, A) \in [\kappa]^{<\kappa} \times F$. The order is defined by $(a, A) \leq (b, B)$ if and only if

(1) $b \sqsubseteq a$.
(2) $A \subseteq B$.
(3) $a \setminus b \subseteq B$

This forcing is κ-closed, and κ^+-cc. It adds a set X which is a $\subseteq^*$-lower bound for the filter F.

The focus of this section is the *iteration of Mathias forcings based on* $\mathbb{D} = (D, \leq_D)$ associated with a sequence $\langle \dot{U}_{<d}, \dot{U}_d : d \in D\rangle$. We say that

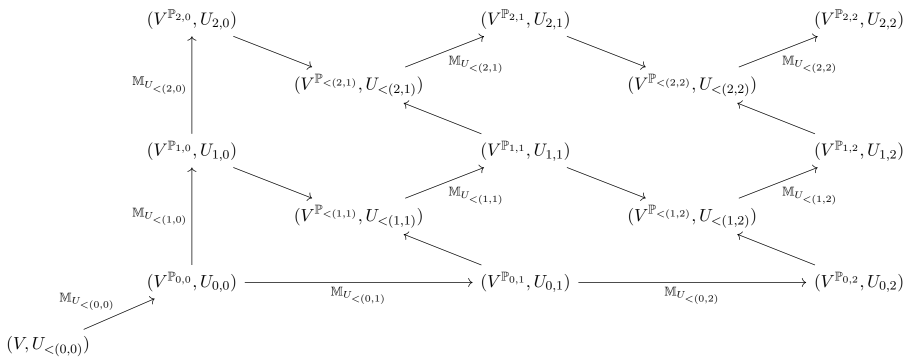

FIGURE 1. Models and ultrafilters of the $\lambda_0 \times \lambda_1$-based iteration

$\langle \dot{U}_{<d}, \dot{U}_d : d \in D\rangle$ is *suitable* if there is an iterated forcing $\langle \mathbb{P}_{<d}, \dot{\mathbb{Q}}_d \mid d \in D\rangle$ based on $\mathbb{D}$, satisfying the following additional requirements:

I. $\dot{U}_{<d}$ is an $\mathbb{P}_{<d}$-name for an ultrafilter on ω that extends $\bigcup_{e<_D d} \dot{U}_e$.

II. $\dot{\mathbb{Q}}_d$ is a $\mathbb{P}_{<d}$-name for the Mathias forcing $\mathbb{M}(\dot{U}_{<d})$.

III. $\dot{U}_d$ is an $\mathbb{P}_{\leq d}$-name for an ultrafilter extending $\dot{U}_{<d} \cup \{\dot{B}_d\}$ where $\dot{B}_d = \{(p, \check{\alpha}) \mid p \restriction D_{<d} \Vdash$ "$\check{\alpha}$ appears in the stem of $p(d)$"$\}$ is the canonical name for an $\dot{\mathbb{Q}}_d$-generic set.

Given $\mathbb{D}$ and a suitable sequence $\langle \dot{U}_{<d}, \dot{U}_d : d \in D\rangle$, the associated iteration of Mathias forcings based on $\mathbb{D}$ is the iteration $\langle \mathbb{P}_{<d}, \dot{\mathbb{Q}}_d : d \in D\rangle$ which is uniquely determined by the choice of $\langle \dot{U}_{<d}, \dot{U}_d : d \in D\rangle$.

Fix such a suitable sequence $\langle \dot{U}_{<d}, \dot{U}_d : d \in D\rangle$ and denote by $\mathbb{M}_D$ the resulting partial order $\mathbb{P}_{\mathbb{D}}$.

Example 3.3. Let λ_0 and λ_1 be regular uncountable cardinals. Let $(D, \leq_D) = (\lambda_0, \leq) \times (\lambda_1, \leq)$, so $\mathbb{D}$ is the directed partial order on $\lambda_0 \times \lambda_1$ given by

$$(\alpha_0, \alpha_1) \leq_D (\beta_0, \beta_1) \iff \alpha_0 \leq \beta_0 \wedge \alpha_1 \leq \beta_1.$$

We will eventually prove (Corollary 3.8) that for such $\mathbb{D}$, it is possible to construct a suitable sequence $\langle \dot{U}_{<d}, \dot{U}_d \mid d \in D\rangle$.

Remark 3.4. If $e <_D d$ then

$$\dot{U}_{<e} \subseteq \dot{U}_e \subseteq \dot{U}_{<d} \subseteq \dot{U}_d$$

is a tower of ultrafilters of the models

$$V^{\mathbb{P}_{<e}} \subseteq V^{\mathbb{P}_e} \subseteq V^{\mathbb{P}_{<d}} \subseteq V^{\mathbb{P}_d}$$

Indeed, $\dot{U}_{<e} \subseteq \dot{U}_e \subseteq \dot{U}_{<d}$. Since the generic set $\dot{B}_d$ added by $\dot{\mathbb{Q}}_d$ diagonalizes $\dot{U}_{<d}$, and $\dot{B}_d \in \dot{U}_d$, it follows that $\dot{U}_{<d} \subseteq \dot{U}_d$.

Lemma 3.5. *Assume that $\mathbb{D}$ is well-founded. Then the set of conditions $p \in \mathbb{M}_{\mathbb{D}}$ such that for all $d \in \mathrm{Supp}(p)$, $p_d = (\check{\tau}_d, \dot{A}_d)$ is dense.*

Proof. This is an easy induction which uses the fact that $\mathbb{M}_{\mathbb{D}}$ has finite support. □

Lemma 3.6. *For every regular uncountable cardinal λ, $\mathbb{M}_{\mathbb{D}}$ is λ-Knaster.*

Proof. This is a straightforward application of the previous lemma combined with the Δ-system lemma and the fact that Mathias forcing is σ-centered. □

Let $G \subseteq \mathbb{M}_{\mathbb{D}}$ be a generic. For every $d \in D$, denote $B_d = (\dot{B}_d)_G$.

Theorem 3.7. $(\{B_d \mid d \in D\}, \supseteq^*)$ *is isomorphic to D.*

Proof. Let us begin observing that if $e <_D d$ then $B_d \subseteq^* B_e$. To see this, note that $B_e \in U_e \subseteq U_{<d}$ by Remark 3.4. Moreover B_d diagonalizes $U_{<d}$, and so $B_d \subseteq^* B_e$.

Now we will show that if $e \not\leq_D d$, then $B_d \not\subseteq^* B_e$. Let $D' = D_{\leq d} \cup D_{<e}$. Since D' is downwards closed, G projects onto a $\mathbb{M}_{\mathbb{D}'}$-generic G' by Lemma 3.1. Towards a contradiction, suppose there is a condition $p \in \mathbb{P}_{D' \cup \{e\}}$ forcing "$\dot{B}_d \setminus \check{m} \subseteq \dot{B}_e$". (Notice that $\dot{B}_d$ is a $\mathbb{P}_{D'}$-name). Extend $p \restriction D'$ to a condition $p' \in \mathbb{P}_{D'}$ which forces for some $k < \omega$ that "$\check{k} \in \dot{B}_d \setminus \max(\check{s} \cup \{\check{m}\})$", where $p(e) = (\check{s}, \dot{A})$. Let $p^* \in \mathbb{P}_{D' \cup \{e\}}$ be the condition defined by $p^* \restriction D' = p'$ and $p^*(e) = \langle \check{s}, \dot{A} \setminus (\check{k}+1)\rangle$ and notice that $p^* \leq_{D' \cup \{e\}} p$. However, p^* forces the negation of "$\check{B}_d \setminus \check{m} \subseteq \dot{B}_e$", which is impossible. □

Corollary 3.8. *Assume that $\mathbb{D} = (D, \leq_D)$ is countably directed and has no maximal elements. Let $G \subseteq \mathbb{M}_{\mathbb{D}}$ be generic. Then $U = \bigcup_{d \in D} U_d \in V[G]$ forms a p-point ultrafilter. Moreover, the ultrafilter U is generated by $\{B_d \mid d \in D\}$.*

Proof. Since each B_d diagonalizes $U_{<d}$, the "moreover" part follows from the equality $\bigcup_{d \in D} U_d = \bigcup_{d \in D} U_{<d}$, which follows from Remark 3.4 and the fact that D has no maximal elements.

To see the first part, note that U has the finite intersection property since it is the directed union of a family of sets with the finite intersection property, namely the U_d's.

To show that U is an ultrafilter, it suffices to show that for every $X \in P(\omega)^{V[G]}$, either $X \in U$ or $\omega \setminus X \in U$. Since each condition in $\mathbb{M}_{\mathbb{D}}$ has finite support, and since $\mathbb{M}_{\mathbb{D}}$ has the ccc, whenever $X \subseteq \omega$, there is a nice name $\dot{X}$ based on maximal antichains A_n for $n < \omega$. Let $\sigma = \bigcup\{\mathrm{Supp}(p) \mid p \in A_n,\ n < \omega\} \subseteq D$. By our assumption, D is countably directed, let $d \in D$ be an upper bound for σ. Thus $X \in V^{\mathbb{P}_{\leq d}}$ and since U_d is an ultrafilter of $V^{\mathbb{P}_{\leq d}}$, either $X \in U_d$ or $\omega \setminus X \in U_d$. Since $U_d \subseteq U$, either $X \in U$ or $\omega \setminus X \in U$. □

Finally, we would like to construct, for a restricted choice of $\mathbb{D}$, a suitable sequence of names $\langle \dot{U}_{<d}, \dot{U}_d : d \in D\rangle$.

Theorem 3.9. *Suppose that $\mathbb{D} = (D, \leq_D)$ is a well-founded degree 2 directed semi-lattice. Then there is a suitable $\langle \dot{U}_{<d}, \dot{U}_d : d \in D\rangle$.*

Proof. We use the well foundedness to define $\dot{U}_{<d}$ and $\dot{U}_d$ recursively. First, we let $\dot{U}_d$ be a canonical name for any uniform ground model ultrafilter on ω whenever d is D-minimal.

By II, once $\dot{U}_{<d}$ was defined, $\dot{B}_d$ is forced to diagonalize $\dot{U}_{<d}$. Hence, the set $\dot{U}_{<d} \cup \{\dot{B}_d\}$ is forced to have the finite intersection property and therefore can be extended (arbitrarily) to an ultrafilter. Let $\dot{U}_d$ be a name for such an ultrafilter. The non-trivial part of the construction is to guarantee condition I, suppose that $\langle \dot{U}_{<e}, \dot{U}_e : e <_D d\rangle$ has been defined.

Lemma 3.10. $\Vdash_{\mathbb{P}_{<d}}$ "$\bigcup_{e<_D d} \dot{U}_e$ *has the finite intersection property".*

Proof. We prove the lemma by induction on the number of sets intersected. Specifically, for each $k \geq 1$ we show that if $e_0, \ldots, e_k <_D d$, and $Y_i \in U_{e_i}$, then $\bigcap_{i=0}^k Y_i$ is infinite. For $k = 1$, let $Y_0 \in U_{e_0}$ and $Y_1 \in U_{e_1}$. Towards contradiction, suppose that $Y_0 \cap Y_1$ is finite and let $p \in \mathbb{P}_{<d}$ force that $\dot{Y}_0 \cap \dot{Y}_1 \setminus l = \emptyset$ for some $l < \omega$. Let $e^m = e_0 \wedge e_1$ be the meet of e_0, e_1. Let $G_{\leq e^m}$ be a V-generic filter for $\mathbb{P}_{\leq e^m}$. Working in $V[G_{\leq e^m}]$, define:

$$\tilde{A} = \Big\{n < \omega \mid \exists q \in \mathbb{P}_{\leq e_0}/G_{\leq e^m},\ q \leq p \restriction D_{\leq e_0},\ q \Vdash_{\mathbb{P}_{\leq e_0}} \check{n} \in \dot{Y}_0\Big\}$$

$$\tilde{B} = \Big\{n < \omega \mid \exists q \in \mathbb{P}_{\leq e_1}/G_{\leq e^m},\ q \leq p \restriction D_{\leq e_1},\ q \Vdash_{\mathbb{P}_{\leq e_1}} \check{n} \in \dot{Y}_1\Big\}$$

We claim that $\tilde{A} \cap \tilde{B} \setminus l = \emptyset$. Otherwise, let $n \in \tilde{A} \cap \tilde{B} \setminus l$ and let q_0, q_1 witness this. Since the conditions q_0, q_1 are in quotients over the same forcing, there is $r \in \mathbb{P}_{\leq e^m}$ such that

$$r \leq q_0 \restriction D_{\leq e^m}, q_1 \restriction D_{\leq e^m} \leq p \restriction D_{\leq e^m}.$$

Define $p^* \in \mathbb{P}_{<d}$ as follows, for $x <_D d$, set

$$p^*(x) = \begin{cases} r(x) & x \leq_D e^m \\ q_0(x) & x \in D_{\leq e_0} \setminus D_{\leq e^m} \\ q_1(x) & x \in D_{\leq e_1} \setminus D_{\leq e^m} \\ p(x) & \text{otherwise} \end{cases}.$$

Note that there is no conflict in the definition since if $x \leq_D e_0$ and $x \leq_D e_1$ then $x \leq_D e_0 \wedge e_1 = e^m$. We conclude that $p^* \leq p$ and $p^* \Vdash_{\mathbb{P}_{<d}} n \in \dot{Y}_0 \cap \dot{Y}_1 \setminus l$, contradiction.

However, by Remark 3.4, $\tilde{A} \in U_{e_0} \cap V[G_{\leq e^m}] = U_{e^m}$ and $\tilde{B} \in U_{e_1} \cap V[G_{\leq e^m}] = U_{e^m}$. Thus, $\tilde{A}, \tilde{B} \in U_{e^m}$, producing a contradiction to the choice of U_{e^m} being a uniform ultrafilter.

Now for the general case, suppose that $k \geq 2$. If there are $i \neq j$ such that e_i and e_j are $<_D$-comparable, then we may reduce the $k+1$ sets $Y_0, .., Y_k$ to k sets and apply the induction hypothesis for $k-1$. Suppose that $e_0, ..., e_k$ are pairwise incomparable. Since D has degree 2, there are $i \neq j$ and

$e^M <_D d$ such that $e_i, e_j \leq_D e^M$. Also, since e_i and e_j are incomparable, $e_i, e_j <_D e^M$. Therefore, $U_{e_i}, U_{e_j} \subseteq U_{<e^M}$. It follows that $B_{e^M} \in U_{e^M}$ is $\subseteq^*$-below both Y_i and Y_j. Replacing Y_i, Y_j by B_{e^M}, we may now apply the induction hypothesis for $k-1$ elements. This concludes the inductive proof. □

□

Corollary 3.11. *Suppose that $\mathbb{D}$ is a well-founded, degree 2, countably directed semi-lattice, with no maximal elements, then there is a ccc forcing extension in which there is a uniform simple $P_{\mathbb{D}}$-point ultrafilter U.*

Corollary 3.12. *Let λ_1, λ_2 be any cardinals of uncountable cofinality. There is a ccc forcing extension in which there is a uniform $P_{\lambda_1 \times \lambda_2}$-point ultrafilter.*

Corollary 3.13. *Let $\mathbb{D}$ be a well-founded countably-directed two-dimensional order with no maximal elements. Then there is a ccc forcing extension in which there is a uniform $P_{\mathbb{D}}$-point ultrafilter.*

Proof. By Corollary 2.10, $\mathbb{D}$ can be cofinally embedded into a degree-two subset A of a product of cardinals. Also since $\mathbb{D}$ is countably directed and has no maximal elements, so does A. Now we can apply Corollary 3.11 to force an ultrafilter U which is a simple P_A-point. Since $\mathbb{D}$ cofinally embeds into A, by Proposition 2.1 U is a simple $P_{\mathbb{D}}$-point. □

4. At a measurable cardinal

In this section, we produce examples of measures, i.e. normal ultrafilters, on a measurable cardinal κ which have complicated yet controlled cofinal structures. As in Section 3 the idea is to produce a measure U such that some poset with a complicated Tukey structure is embedded cofinally into $(U, \supseteq^*)$, and we produce U by a nonlinear iteration of a version of ultrafilter Mathias forcing. However, we will use a different method to choose the forcing posets used at each stage, which permits us to embed a more general class of posets into $(U, \supseteq^*)$. For definiteness we will show how to embed the product of a finite increasing sequence of regular cardinals greater than κ.

To illustrate the main idea in a simple setting, suppose that κ is a Laver indestructible supercompact cardinal and consider a linear iteration of ultrafilter Mathias forcing at κ with $< \kappa$-supports, where at stage α we use a measure $U_\alpha \in V[G_\alpha]$ to define the next step in the iteration: there is no problem in choosing some U_α because the forcing so far is κ-directed closed. In the final model $V[G]$ there is of course a measure U, but *a priori* we have no reason to believe that the set B_α added at stage α is in U, or even that $U_\alpha \subseteq U$.

This problem was first solved by Džamonja and Shelah [22], where the rough idea is that we force at α with the disjoint sum of all measures which exist in $V[G_\alpha]$. This leads to some technical complications in the iteration, which we can avoid using an idea suggested originally by Magidor: the idea is to choose U_α using some diamond principle, and organize the construction so that whenever $U_\alpha \subseteq U$ (which happens frequently) we have $B_\alpha \in U$. Running the iteration for a large enough number of steps, we may produce measures which are generated by arbitrarily long $\subseteq^*$-decreasing sequences. The diamond idea appears in a paper by Cummings, Džamonja, Magidor, Morgan and Shelah [18] albeit in a more complex setting.

For technical reasons we will be using "filter Mathias forcing" (see Definition 3.2). If F is a κ-complete filter, the filter Mathias poset is still κ-directed closed and κ^+-cc.

Let $\mathbb{Q}$ be a forcing poset and let λ be a cardinal. It will be convenient to specify exactly what we mean by a *canonical* $\mathbb{Q}$*-name* for a subset of λ or a family of subsets of λ. A canonical $\mathbb{Q}$-name for a subset of λ is a name $\dot\tau$ such that $\dot\tau \subseteq \mathbb{Q} \times \{\check\alpha : \alpha < \lambda\}$, and $\{q : (q, \check\alpha) \in \dot\tau\}$ is an antichain for all α. A canonical name for a family of subsets of λ is a name $\dot\sigma$ such that $\dot\sigma \subseteq \mathbb{Q} \times \{\dot\tau : \dot\tau$ is a canonical name for a subset of $\lambda\}$, and $\{q : (q, \dot\tau) \in \dot\sigma\}$ is an antichain for all $\dot\tau$. It is easy to see that any name for a subset of λ or a family of subsets of λ is equivalent to a canonical name.

Let $V_0 \models$ "GCH and κ is supercompact", and let $V = V_0[L]$ where $\mathbb{L}$ is the standard Laver preparation to make κ indestructible under κ-directed forcing. Note that GCH holds in V for all cardinals δ with $\delta \geq \kappa$.

We work in V until further notice. Let $n < \omega$, let $(\lambda_i)_{0 \leq i \leq n}$ be a strictly increasing sequence of regular cardinals with $\kappa < \lambda_0$, and let $I = \prod_{i \leq n} \lambda_i$. We will define a non-linear iteration $\mathbb{P}$ of ultrafilter Mathias forcing indexed by I, arranging that $\mathbb{P}$ is κ-directed closed and κ^+-cc.

We will also assume that the diamond principle $\diamondsuit_{\lambda_n}(\lambda_n \cap \mathrm{cof}(\kappa^+))$ holds in V. This is easy to arrange: prepare V_0 so that this form of diamond holds there, and then use standard arguments about the preservation of diamond principles by forcing to show that it still holds in V.[1] We will use a form of diamond which is adapted to guessing subsets of H_{λ_n}, which is possible because $|H_{\lambda_n}| = \lambda_n$. To be precise we will assume that there are sequences $(A_\alpha)_{\alpha \in \lambda_n \cap \mathrm{cof}(\kappa^+)}$ and $(M_\alpha)_{\alpha < \lambda_n}$ such that

(1) $M_\alpha \prec H_{\lambda_n}$ with $|M_\alpha| < \lambda_n$.
(2) $(M_\alpha)_{\alpha < \lambda_n}$ is continuous and $\bigcup_{\alpha < \lambda_n} M_\alpha = H_{\lambda_n}$.
(3) $A_\alpha \subseteq M_\alpha$, and for every $A \subseteq H_{\lambda_n}$ there are stationarily many $\alpha \in \lambda_n \cap \mathrm{cof}(\kappa^+)$ such that $A \cap M_\alpha = A_\alpha$.

The choice of the sequence (M_α) is not important, since such sequences always exist and any two such sequences agree on a club subset of λ_n.

As before, we will order I with the product coordinatewise ordering, that is to say $(\eta_i)_{i \leq n} \leq (\zeta_i)_{i \leq n}$ if and only if $\eta_i \leq \zeta_i$ for all $i \leq n$. For $a =$

[1]Our GCH hypothesis actually implies right away this diamond principle at many, but not all, instances. For example, by Shelah [43] it holds for every successor cardinal $\lambda_n = \chi^+$ as long as $\mathrm{cf}\chi \neq \kappa^+$. By an unpublished result of the third author, it holds for every λ_n inaccessible, as λ_n is above a supercompact cardinal.

$(\alpha_1, \dots, \alpha_n) \in I$, recall that we denoted

$$I_{<a} = \{b \in I : b < a\} = (\alpha_0 + 1) \times ... \times (\alpha_n + 1) \setminus \{a\}.$$

As usual $\mathbb{P}_a$ is defined by recursion on a. Part of the definition will involve choosing a $\mathbb{P}_a$-name $\dot{F}_a$ for a κ-complete uniform filter on κ in $V[P_a]$, but we postpone the exact definition of $\dot{F}_a$ for the moment.

A condition $p \in \mathbb{P}_b$ is a partial function p such that:

- $\operatorname{dom}(p) \subseteq I_{<b}$.
- $|\operatorname{dom}(p)| < \kappa$.
- For every $a \in \operatorname{dom}(p)$, $p(a)$ is a $\mathbb{P}_a$-name for a condition in $\mathbb{Q}_a$, where $\mathbb{Q}_a$ is the filter Mathias forcing defined in $V^{\mathbb{P}_a}$ from the filter $\dot{F}_a$.

For conditions $p, q \in \mathbb{P}_b$, $q \leq p$ if and only if:

- $\operatorname{dom}(p) \subseteq \operatorname{dom}(q)$.
- For all $a \in \operatorname{dom}(p)$, $q \restriction I_{<a} \Vdash_{\mathbb{P}_a} q(a) \leq_{\mathbb{Q}_a} p(a)$.

The definition makes sense because (by an easy induction) for all $a < b$ we have that $\mathbb{P}_a$ is a complete subposet of $\mathbb{P}_b$, and $q \mapsto q \restriction I_{<a}$ is a projection from $\mathbb{P}_b$ to $\mathbb{P}_a$. It is easy to verify that $\mathbb{P}_a$ is κ-directed closed. For κ^+-cc we first verify that the set of p such that $p \restriction I_{<a}$ determines the stem of $p(a)$ for all $a \in \operatorname{dom}(p)$ is dense, and then use the Δ-system lemma, keeping in mind that $\kappa^{<\kappa} = \kappa$.

Let $\mathbb{P} = \bigcup_{b \in I} \mathbb{P}_b$, be the direct limit of $\langle \mathbb{P}_b \mid b \in I \rangle$. For any downwards closed subset J of I, let $\mathbb{P}_J = \{p \in \mathbb{P} : \operatorname{dom}(p) \subseteq J\}$. Then $\mathbb{P}_J$ is a complete subposet of $\mathbb{P}$, and $q \mapsto q \restriction J$ is a projection from $\mathbb{P}$ to $\mathbb{P}_J$ (the proof is identical to Lemma 3.1). In these notations, $\mathbb{P}_a = \mathbb{P}_{I_{<a}}$.

We still owe the definition of a $\mathbb{P}_a$-name $\dot{F}_a$ for a κ-complete uniform filter on κ. For $\beta < \lambda_n$, we say that β is an *active stage* if $\beta \in \lambda_n \cap \operatorname{cof}(\kappa^+)$ and A_β is a canonical $\mathbb{P}_{\lambda_0 \times ... \times \lambda_{n-1} \times \beta}$-name for a measure on κ. Let $a = (\alpha_0, \dots, \alpha_{n-1}, \beta)$ and assume for the moment that β is an active stage.

Working in V, let F^0_a be the set of canonical $\mathbb{P}_{(\alpha_0+1) \times ... \times (\alpha_{n-1}+1) \times \beta}$-names $\dot{A}$ for subsets of κ such that $\Vdash^V_{\mathbb{P}_{\lambda_0 \times ... \times \lambda_{n-1} \times \beta}} \dot{A} \in A_\beta$. This is reasonable because $(\alpha_0 + 1) \times \dots \times (\alpha_{n-1} + 1) \times \beta$ is a downwards closed subset of $\lambda_0 \times \dots \times \lambda_{n-1} \times \beta$, so that $\dot{A}$ makes sense as a $\mathbb{P}_{\lambda_0 \times ... \times \lambda_{n-1} \times \beta}$-name for a subset on κ.

Claim 4.1. *Let G_a be $\mathbb{P}_a$-generic and let $F^1_a = \{\dot{A}[G_a] : \dot{A} \in F^0_a\}$. Then $V[G_a] \models$ "F^1_a generates a κ-complete uniform filter".*

Proof. Force to prolong G_a to G such that $G \restriction I_{<a}$ is $\mathbb{P}_{\downarrow a}$-generic, and let $W = A_\beta[G \restriction \lambda_0 \times \dots \times \lambda_{n-1} \times \beta]$. Then $F^1_a \subseteq W$ and $V[G \restriction \lambda_0 \times \dots \times \lambda_{n-1} \times \beta] \models$ "W is a measure on κ", and the conclusion follows easily. □

If $a = (\alpha_0, \dots, \alpha_{n-1}, \beta)$ and β is an active stage, we let $\mathcal{X}_a = \{B_b \mid b = (\beta_1, ..., \beta_{n-1}, \beta) < a\}$, where B_b is the Mathias generic set added by the component $\mathbb{Q}_b$ for the filter F_b. Note that the right-most coordinate is fixed in the definition of $\mathcal{X}_a$. If $F^1_a \cup \mathcal{X}_a$ generates a κ-complete filter, let F_a be the κ-complete uniform filter generated in $V[G_a]$ by $F^1_a \cup \mathcal{X}_a$. If β is not an active stage or if $F^1_a \cup \mathcal{X}_a$ does not generate a κ-complete filter, then let F_a be the tail filter on κ.

We record a few easy observations, where we make repeated use of the κ^+-cc and the regularity assumption on the λ_i's to analyze how subsets of κ will appear in the generic extension.

(1) Every canonical $\mathbb{P}$-name for a subset of κ is a canonical $\mathbb{P}_a$-name for some $a \in I$. It follows that if G is $\mathbb{P}$-generic then $P(\kappa) \cap V[G] = \bigcup_{a \in I} P(\kappa) \cap V[G_a]$, where G_a is the projection of G to $\mathbb{P}_a$.
(2) Similarly if $\beta < \lambda_n$ and $\mathrm{cf}(\beta) > \kappa$, then $P(\kappa) \cap V[G_{\prod_{i<n} \lambda_i \times \beta}] = \bigcup_{\alpha < \beta} P(\kappa) \cap V[G_{\prod_{i<n} \lambda_i \times \alpha}]$.

We will prove:

Theorem 4.2. *If G is $\mathbb{P}$-generic then there exist in $V[G]$ a measure U on κ and a family of sets $(B_a)_{a \in I}$ such that*

(1) *$B_a \in U$ for all $a \in I$.*
(2) *For all $a, b \in I$, $B_b \subseteq^* B_a$ if and only if $a \leq b$.*
(3) *For every $Y \in U$ there is $a \in I$ such that $B_a \subseteq^* Y$.*

In particular, U is a simple P_I-point.

Proof of Theorem 4.2. Working in V_0 we fix $j : V_0 \to M_0$ such that j witnesses that κ is λ_n-supercompact, and $j(\mathbb{L}) \simeq \mathbb{L} * \mathbb{P} * \mathbb{R}$ where the first point in the support of the tail iteration $\mathbb{R}$ is greater than λ_n. This ensures that:

(1) $\Vdash^{M_0}_{\mathbb{L} * \mathbb{P}}$ "$\mathbb{R} * j(\mathbb{P})$ is λ_n^+-closed".
(2) $V[G] \models {}^{\lambda_n} M_0[L][G] \subseteq M_0[L][G]$
(3) $V[G] \models$ "$\mathbb{R} * j(\mathbb{P})$ is λ_n^+-closed".

It is straightforward to lift $j : V_0 \to M_0$ to obtain $j : V = V_0[L] \to M_0[L * G * H]$ for any H which is $\mathbb{R}$-generic over $V[G]$. To further lift j onto $V[G]$ we would need a $j(\mathbb{P})$-generic object S such that $j[G] \subseteq S$. It is worth noting that, since conditions in $\mathbb{P}$ have supports of size less than κ, the support of a condition in $j[G]$ will be contained in $j[I]$.

Let $\prec$ be the lexicographic ordering on I, so that:

- $\prec$ is a well-ordering of I with order type λ_n.
- $\prec$ extends $<$.

Let $(a_i)_{i < \lambda_n}$ enumerate I in $\prec$-increasing order.

Working in $V[G]$, we will construct a decreasing sequence $(r_i, s_i)_{i < \lambda_n}$ of conditions in $\mathbb{R} * j(\mathbb{P})$. We will arrange that:

- The support of s_i is contained in $j(\{a_\eta : \eta < i\})$, so that in particular $j(a_\zeta)$ is not in the support of s_i for $\zeta \geq i$.
- $s_i \leq j[G_{a_i}]$, so that (r_i, s_i) forces that j can be lifted onto $V[G_{a_i}]$.

We note that s_i may not actually be a condition in $j(\mathbb{P}_{a_i})$, the point is that the projection of s_i to $j(\mathbb{P}_{a_i})$ forces that the $j(\mathbb{P}_{a_i})$-generic object contains $j[G_{a_i}]$.

For $i = 0$, where we note that $a_0 = (0, 0, \dots, 0)$, let (r_0, s_0) be the trivial condition. For i limit let (r_i, s_i) be a lower bound for $(r_j, s_j)_{j<i}$, taking care that the support of s_i is the union of the supports of s_j for $j < i$.

To construct (r_{i+1}, s_{i+1}), we force below (r_i, s_i) and construct $j : V[G_{a_i}] \to M_0[L][G][H][S']$ where $H * S'$ is $\mathbb{R} * j(\mathbb{P}_{a_i})$-generic. Recall that we defined $\mathbb{Q}_{a_i}$ from a κ-complete uniform filter $F_{a_i} \in V[G_{a_i}]$, and that the a_i-component of G gives a subset B_{a_i} of κ which is $\mathbb{Q}_{a_i}$-generic.

Let s'_i be the projection (i.e. restriction) of s_i to $j(\mathbb{P}_{a_i})$. There are now two cases:

1. $(r_i, s'_i) \Vdash^{M_0[L][G]}_{\mathbb{R} * j(\mathbb{P}_{a_i})} \kappa \in j(E)$ for every $E \in F_{a_i}$. In this case we define r_{i+1} and s_{i+1} as follows:
   (a) $r_{i+1} = r_i$.
   (b) $\mathrm{dom}(s_{i+1}) = \mathrm{dom}(s_i) \cup \{j(a_i)\}$.
   (c) $s_{i+1} \restriction \mathrm{dom}(s_i) = s_i$.
   (d) $s_{i+1}(j(a_i)) = (B_{a_i} \cup \{\kappa\}, \bigcap j[F_{a_i}])$.
2. There is $E \in F_{a_i}$ such that $(r_i, s'_i) \not\Vdash^{M_0[L][G]}_{\mathbb{R} * j(\mathbb{P}_{a_i})} \kappa \in j(E)$. In this case we choose such an $E_i = E$ and find $(r^*_i, s^*_i) \in \mathbb{R} * j(\mathbb{P}_{a_i})$ such that $(r^*_i, s^*_i) \le (r_i, s'_i)$ and $(r^*_i, s^*_i) \Vdash \kappa \notin j(E_i)$. We then define:
   (a) $r_{i+1} = r^*_i$.
   (b) $\mathrm{dom}(s_{i+1}) = \mathrm{dom}(s^*_i) \cup (\mathrm{dom}(s_i) \setminus \mathrm{dom}(s^*_i)) \cup \{j(a_i)\}$.
   (c) $s_{i+1} \restriction \mathrm{dom}(s^*_i) = s^*_i$.
   (d) $s_{i+1} \restriction (\mathrm{dom}(s_i) \setminus \mathrm{dom}(s^*_i)) = s_i \restriction (\mathrm{dom}(s_i) \setminus \mathrm{dom}(s^*_i))$.
   (e) $s_{i+1}(j(a_i)) = (B_{a_i}, \bigcap j[F_{a_i}])$.

Let us check that we have maintained the hypotheses. The main points are:

1. $j(F_{a_i})$ is $j(\kappa)$-complete and $j[F_{a_i}] \in M_0[L][G][H][S']$ so that $\bigcap j[F_{a_i}] \in j(F_{a_i})$ and we constructed legitimate conditions in $j(\mathbb{Q}_{a_i})$.
2. In either case of the construction, $\mathrm{dom}(s_{i+1}) \subseteq \mathrm{dom}(s_i) \cup j(I_{<a_i}) \cup \{j(a_i)\}$. Since $I_{<a_i} \subseteq \{a_\eta : \eta < i\}$, it follows that $\mathrm{dom}(s_{i+1}) \subseteq j(\{a_\zeta : \zeta \le i\})$.
3. The filter added by G at a_i is the set of pairs (s, E) where $E \in F_a$, s is an initial segment of B_{a_i} and $B_{a_i} \setminus s \subseteq E$. Since $j((s, E)) = (s, j(E))$, it is easy to see that it is forced that $(B_{a_i}, \bigcap j[F_{a_i}]) \le (s, j(E))$. In the first case (r_i, s_i) forces that $(B_{a_i} \cup \{\kappa\}, \bigcap j[F_{a_i}]) \le (s, j(E))$.

When the construction of $(r_i, s_i)_{i<\lambda_n}$ is done, we choose (r', s') to be a lower bound and note that (r', s') forces that j can be lifted onto $V[G]$. Recall that every subset of κ in $V[G]$ lies in $V[G_a]$ for some $a \in I$, and observe that if $X = \{\dot{A} : \text{For some } a \in I,\ \dot{A} \text{ is a canonical } \mathbb{P}_a\text{-name for a subset of } \kappa\}$ then $|X| = \lambda_n$. Using the closure of $\mathbb{R} * j(\mathbb{P})$ again, we find $(r, s) \le (r', s')$ such that (r, s) decides $\kappa \in j(\dot{A})$ for every $\dot{A} \in X$. It is now easy to see that if $U = \{\dot{A}[G] : \dot{A} \in X \text{ and } (r, s) \Vdash \kappa \in j(\dot{A})\}$ then $V[G] \models$ "U is a measure on κ".

Let $\dot{U}$ be a canonical name for the measure U produced by this construction, and for $\beta < \lambda_n$ let

$$\dot{U}_\beta = \{(q, \dot{\tau}) \in \dot{U} : q \in \mathbb{P}_{\prod_{i<n} \lambda_i \times \beta} \text{ and } \dot{\tau} \text{ is a canonical } \mathbb{P}_{\prod_{i<n} \lambda_i \times \beta}\text{-name for a subset of } \kappa\}$$

Then:

- $\dot{U} \subseteq H_{\lambda_n}$.
- $\dot{U} = \bigcup_{\beta<\lambda_n} \dot{U}_\beta$, and $\langle \dot{U}_\beta \mid \beta < \lambda \rangle$ is continuous at points of cofinality κ^+.
- For almost every $\beta \in \lambda_n \cap \mathrm{cof}(\kappa^+)$, $\dot{U}_\beta$ is a name for a measure on κ, and is forced to equal $U \cap V[G_{\prod_{i<n} \lambda_i \times \beta}]$.
- If $S = \{\beta \in \lambda_n \cap \mathrm{cof}(\kappa^+) : A_\beta = \dot{U}_\beta\}$ then S is stationary in λ_n, and every $\beta \in S$ is an "active stage" in the definition of $\mathbb{P}$.

Let $I^* = \lambda_0 \times \dots \lambda_{n-1} \times S$, and note that $(I^*, <)$ is isomorphic to $(I, <)$.

Claim 4.3. *For every $a \in I^*$, $B_a \in U$.*

Proof. Let $a = a_i = (\alpha_0, \dots \alpha_{n-1}, \beta)$, and consider the construction of (r_{i+1}, s_{i+1}). The proof is by induction on a. The key point is that we must be in case 1 of the construction, so assume for a contradiction that we are in case 2. Since $(r, s) \leq (r_{i+1}, s_{i+1})$, this means that there is $E \in F_{a_i}$ such that $E \notin U$. Since U is κ-complete and since F_{a_i} is the κ-complete filter generated by $F^1_a \cup \mathcal{X}_a$, either there is $Y \in A_\beta[G_{\lambda_0 \times \dots \lambda_{n-1} \times \beta}] \setminus U$ or $B_{a'} \notin U$ for some $a' = (\alpha'_1, ..., \alpha'_{n-1}, \beta) < a$. Note that $a' \in I^*$ as well, hence, by the induction hypothesis, $B_{a'} \in U$. In the other case, since β is an active stage, $A_\beta = \dot{U}_\beta$, and thus $Y \in U \cap V[G_{\prod_{i<n} \lambda_i \times \beta}]$. In either case, we reach a contradiction.

Since we are in case 1, $(r_{i+1}, s_{i+1}) \Vdash \kappa \in j(B_a)$. So $(r, s) \Vdash \kappa \in j(B_a)$, and therefore $B_a \in U$. □

As a corollary, we conclude that whenever β is an active stage of the iteration, for every $a = (\alpha_1, ..., \alpha_{n-1}, \beta)$, $F^1_a \cup \mathcal{X}_a \subseteq U$ and therefore generates a κ-complete ultrafilter. By the definition of F_a, this means that $F_1 \cup \mathcal{X}_a \subseteq F_a$.

Claim 4.4. *Let $a, a' \in I^*$ where $a < a'$. Then $B_a \in F_{a'}$.*

Proof. Let $a = (\alpha_0, \dots \alpha_{n-1}, \beta)$ and and $a' = (\alpha'_0, \dots \alpha'_{n-1}, \beta')$. If $\beta = \beta'$, then $B_a \in \mathcal{X}_{a'}$ and we have already seen that $\mathcal{X}_{a'} \subseteq F_{a'}$. Assume that $\beta' < \beta$, then clearly $B_a \in V[G_{(\alpha_1+1)\times \dots \times (\alpha_{n-1}+1) \times \beta'}]$. Let $\dot{B}_a$ be a canonical $\mathbb{P}_{a'}$-name for B_a. We need to show that $\Vdash^V_{\mathbb{P}_{\lambda_0 \times \dots \times \lambda_{n-1} \times \beta'}} \dot{B}_a \in A_{\beta'}$. Let $G_{\lambda_0 \times \dots \times \lambda_{n-1} \times \beta'}$ be an arbitrary generic for $\mathbb{P}_{\lambda_0 \times \dots \times \lambda_{n-1} \times \beta'}$, and force to prolong it to a $\mathbb{P}$-generic G. Then by Claim 4.3

$$B_a \in U \cap V[G_{\lambda_0 \times \dots \times \lambda_{n-1} \times \beta'}] = U_{\beta'}[G_{\lambda_0 \times \dots \times \lambda_{n-1} \times \beta'}] = A_{\beta'}[G_{\lambda_0 \times \dots \times \lambda_{n-1} \times \beta'}]$$

Hence, by definition, $\dot{B}_a \in F^0_{a'}$ and $B_a \in F^1_{a'} \subseteq F_{a'}$. □

Claim 4.5. *Let $a, a' \in I^*$ Then $B_{a'} \subseteq^* B_a$ if and only if $a \leq a'$.*

Proof. If $a = a'$ then the implication is clear. If $a < a'$ then $B_a \in F_{a'}$ by Claim 4.4. Since $B_{a'}$ is generic for the filter Mathias forcing defined from $F_{a'}$, $B_{a'} \subseteq^* B_a$.

Now let $a \not\leq a'$. We work in the dense subset of $\mathbb{P}$ where $p \restriction I_{<b}$ decides the stem s^p_b of the condition $p(b)$ for all $b \in \text{dom}(p)$, and write $p(b) = (s^p_b, \dot{A}^p_b)$.

Assume for a contradiction that $p \in G$ and $p \Vdash B_{a'} \setminus \zeta \subseteq B_a$ for some $\zeta < \kappa$. We may assume that $a, a' \in \text{dom}(p)$. Now let $I_{\leq a'} = I_{<a'} \cup \{a'\}$, and note that $a \notin I_{\leq a'}$. Extend $p \restriction I_{\leq a'}$ to some $q \in \mathbb{P}_{I_{\leq a'}}$ so that $\gamma \in s^q_{a'}$ for some $\gamma > \zeta, \max(s^p_a)$. Let $p_1 = p \cup q$ and note that $p_1(a) = p(a) = (s^p_a, \dot{A}^p_a)$. Extend p_1 to p_2 such that $p_2(a) = (s^p_a, \dot{A}^p_a \setminus (\gamma + 1))$. Then $p_2 \Vdash$ "$\gamma \in \dot{B}_{a'} \setminus \zeta$ and $\gamma \notin B_a$" for an immediate contradiction. □

Claim 4.6. *In $V[G]$, $\{B_a : a \in I^*\}$ forms a basis for U.*

Proof. Let $Y \in V[G]$ with $Y \in U$. Then $Y \in V[G_a]$ for some $a \in I^*$. Arguing as in the proof of Claim 4.4, there is $b \in I^*$ such that $Y \in F_b$, so that $B_b \subseteq^* Y$. □

This concludes the proof of Theorem 4.2.

□

Corollary 4.7. *Relative to a supercompact cardinal, it is consistent that there is a supercompact cardinal κ such that for any regular cardinals $\kappa < \lambda_0 < ... < \lambda_n$, there is a κ-directed closed, κ^+-cc forcing extension where there is a normal ultrafilter over κ which is a simple $P_{\lambda_0 \times ... \times \lambda_n}$-point.*

Remark 4.8. The method presented in this section can be modified to obtain a model where there a $P_{\mathbb{D} \times \lambda}$-point ultrafilter on κ for $\mathbb{D} = (D, \leq_D)$ and λ such that:

(1) $|D| < \lambda = cf(\lambda)$.
(2) $\mathbb{D}$ is κ^+-directed and well-founded.

The only modifications in the proof above are cosmetic:

- work with the order $(d, \beta) \leq (d', \beta')$ if and only if $d \leq_D d' \wedge \beta \leq \beta'$.
- replace $\prod_{0 \leq i \leq n-1} \lambda_i$ by $\mathbb{D}$.
- An ordinal $\beta < \lambda$ is active if A_β guesses a $\mathbb{P}_{\mathbb{D} \times \beta}$-name for a measure. Note that $\mathbb{D} \times \beta$ consistes of pairs (d, γ) such that $\gamma < \beta$. Consider a pair $a = (d, \beta)$, F^0_a consists of $\mathbb{P}_a$-names (i.e. less or equal in both coordinates) $\dot{A}$ which are forced to be in A_β. Then F^1_a, $\mathcal{X}_a$ and F_a are defined the same way,
- extend the order on $\mathbb{D} \times \lambda$ to a well-ordering of ordertype λ.
- use the κ^+-directedness of $\mathbb{D} \times \lambda$ to analyze as before how sets appear in the intermediate models, and the fact that $|\mathbb{D}| < \lambda$ to run the diamond arguments.

Corollary 4.9. *Relative to the existence of a supercompact cardinal κ, it is consistent that κ is supercompact and for all regular cardinals $\kappa < \lambda_0 <$*

$\lambda_2 < ... < \lambda_n$ there is a κ^+-cc and κ-closed poset forcing that there is a base $\mathcal{C}$ for the club filter Cub_κ such that $(\mathcal{C}, \supseteq^*) \simeq \lambda_0 \times \lambda_2 \times ... \times \lambda_n$.

Proof. Let $\mathcal{B}$ be the base from the previous theorem and let $\mathcal{C}$ be the set of closures of the sets of $\mathcal{B}$. Then since $\mathcal{B}$ generated a normal ultrafilter, $\mathcal{C}$ generates Cub_κ. Clearly, if $X \subseteq^* Y$ then $cl(X) \subseteq^* cl(Y)$. A closer look at the proof of Claim 4.5 will reveal that if $X, Y \in \mathcal{B}$ and $X \not\subseteq^* Y$, then $X \not\subseteq^* cl(Y)$. Indeed, the choice of γ is such that γ is forced to be outside of the closure of $\dot{C}_{\alpha,\beta}$. Therefore, $cl(X) \not\subseteq^* cl(Y)$. □

5. Applications to cofinal types and cardinal characteristics

The following set is also known as the *point-spectrum* which was studied for example in [30, 25, 4, 24].

Definition 5.1. For a directed set $\mathbb{D}$ we define the *Tukey spectrum* of $\mathbb{D}$, $\mathrm{Sp}_T(\mathbb{D})$ to consist of all regular cardinals λ such that $\lambda \leq_T \mathbb{D}$.

Clearly, the Tukey spectrum is an invariant of the Tukey order. An equivalent condition for $\lambda \in \mathrm{Sp}_T(\mathbb{D})$ is to require the existence of a sequence $\langle d_\alpha \mid \alpha < \lambda\rangle$ such that for every $I \in [\lambda]^\lambda$, $\{d_\alpha \mid \alpha \in I\}$ is unbounded in $\mathbb{D}$. For the following fact, see for example [25]:

Fact 5.2.
(1) For any directed sets $\mathbb{P}, \mathbb{Q}$, $\mathrm{Sp}_T(\mathbb{P} \times \mathbb{Q}) = \mathrm{Sp}_T(\mathbb{P}) \cup \mathrm{Sp}_T(\mathbb{Q})$.
(2) If $\mathbb{P}$ is linear, then $\mathrm{Sp}_T(\mathbb{P}) = \{cf(\mathbb{P})\}$.

The main results of the previous sections have the following immediate corollaries:

Corollary 5.3. *Suppose that $\mathbb{D}$ is a well-founded, degree 2, countably directed semi-lattice, with no maximal elements, then there is a ccc forcing extension in which there is an ultrafilter U on ω such that*

$$\mathrm{Sp}_T(U) = \mathrm{Sp}_T(\mathbb{D}).$$

Corollary 5.4. *Relative to the existence of a supercompact cardinal, there is a model V with a supercompact cardinal κ such that for every regular cardinals $\kappa < \lambda_0 < \lambda_1 < ... < \lambda_n$, there is a $<\kappa$-directed closed κ^+-cc forcing extension in which there is a normal ultrafilter U such that:*

$$\mathrm{Sp}_T(U) = \mathrm{Sp}_T(\lambda_0 \times ... \times \lambda_n) = \{\lambda_0, ..., \lambda_n\}.$$

In the previous corollary, $\mathrm{Sp}_T(\mathbb{D})$ is computed in the generic extension, which might be sensitive to forcing. The next proposition shows that no discrepancy arises in the interpretation of the spectrum when the generic extension is a ccc forcing extension. More specifically:

Proposition 5.5. *Let $\mathbb{P}$ be a λ-cc forcing and $\mathbb{D} = (D, \leq_D)$ a countably-directed set. Let $G \subseteq \mathbb{P}$ be a V-generic filter, then*

$$\lambda \in \mathrm{Sp}_T(\mathbb{D})^V \textit{ if and only if } \lambda \in \mathrm{Sp}_T(\mathbb{D})^{V[G]}.$$

Proof. If $\lambda \in \mathrm{Sp}_T(\mathbb{D})^V$, then there is a sequence $\langle d_\alpha \mid \alpha < \lambda\rangle \subseteq D$ such that for any $I \in [\lambda]^\lambda$, $\{d_\alpha \mid \alpha \in I\}$ is unbounded in $\mathbb{D}$. To see that this sequence witnesses that $\lambda \in \mathrm{Sp}_T(\mathbb{D})^{V[G]}$, suppose not, then there is $I \in [\lambda]^\lambda$, $I \in V[G]$ such that $\{d_\alpha \mid \alpha \in I\}$ is bounded by some $d \in D$. Back in V, consider $I^* = \{\alpha < \lambda \mid d_\alpha \leq_D d\}$. Then $I \subseteq I^*$ and therefore $|I^*| = \lambda$. This is a contradiction since $\{d_\alpha \mid \alpha \in I^*\}$ should be unbounded.

For the converse, suppose that $\lambda \in \mathrm{Sp}_T(\mathbb{D})^{V[G]}$. Let $\langle d_\alpha \mid \alpha < \lambda\rangle \subseteq D$ be a sequence in $V[G]$ such that for each index set $I \in [\lambda]^\lambda$ the sequence $\langle d_\alpha \mid \alpha \in I\rangle$ is unbounded. Let $p \in G$ be a condition forcing the above. For each $\alpha < \lambda$ let $\dot{d}_\alpha$ be a $\mathbb{P}$-name such that $(\dot{d}_\alpha)_G = d_\alpha$. For each $\alpha < \lambda$ let $p_\alpha \leq p$ such that $p_\alpha \Vdash_{\mathbb{P}} \dot{d}_\alpha = \check{e}_\alpha$ for some $e_\alpha \in D$. We claim that $\langle e_\alpha \mid \alpha < \lambda\rangle$ witnesses $\lambda \in \mathrm{Sp}(\mathbb{D})^V$. Suppose otherwise, and let $I \in [\lambda]^\lambda$ be such that $\langle e_\alpha \mid \alpha \in I\rangle$ is bounded. To contradict this assumption it suffices to check that $p \Vdash_{\mathbb{P}}$ "$|\{\alpha < \lambda \mid p_\alpha \in \dot{G}\}| = \lambda$". So, let us do it: Suppose towards a contradiction that this was false. Since $\mathbb{P}$ is λ-cc there is $\beta < \lambda$ such that $p \Vdash \{\alpha < \lambda \mid p_\alpha \in \dot{G}\} \subseteq \beta$. Note that this is impossible because p_β itself forces "$p_\beta \in \dot{G}$". □

A related notion is the depth spectrum introduced in [4]:

Definition 5.6. Let $\mathbb{D} = (D, \leq_D)$ be a directed set. A $\mathbb{D}$*-tower of length* λ is a sequence $\langle d_\alpha \mid \alpha < \lambda\rangle$ which is $\leq_D$-increasing and unbounded in $\mathbb{D}$. The *depth spectrum of* $\mathbb{D}$, denoted by $\mathrm{Sp}_{dp}(\mathbb{D})$ consists of all regular cardinals λ such that there is a $\mathbb{D}$-tower of length λ.

The following proposition is straightforward:

Proposition 5.7. *Let* $\mathbb{D}_i = (D_i, \leq_{D_i})$, $i = 1, 2$ *be directed sets. If there is a monotone Tukey reduction* $f : D_1 \to D_2$ *then* $\mathrm{Sp}_{dp}(D_1) \subseteq \mathrm{Sp}_{dp}(D_2)$.

Remark 5.8. If $\mathcal{B}$ is cofinal in $\mathbb{D} = (D, \leq_D)$, then[2], $\mathbb{D} \equiv_T \mathcal{B}$. Moreover, the identity map from $\mathcal{B}$ into D is a monotone Tukey reduction.

Our interest is in orders of the form $(U, \supseteq^*)$ where U is an ultrafilter. Recall that the decomposability spectrum of an ultrafilter U over κ, denoted by $\mathrm{Sp}_{dc}(U)$, is defined as all regular λ such that U is λ-decomposable, that is, there is $f : \kappa \to \lambda$ such that f is unbounded in λ mod U. We have that

$$\mathrm{Sp}_{dc}(U) \setminus \{\kappa\} \subseteq \mathrm{Sp}_{dp}(U) \setminus \{\kappa\} \subseteq \mathrm{Sp}_T(U) \setminus \{\kappa\}$$

The following was asked in [4]:

Question 5.9. Is $\mathrm{Sp}_T(U)$ a convex set of regulars? How about $\mathrm{Sp}_{dp}(U)$?

Using the consistency results from the previous sections and the next proposition we are able to provide a negative answer to this question.

Proposition 5.10. *Suppose that* $\mathcal{F}$ *is a simple* $P_{\lambda_0\times\ldots\times\lambda_n}$*-point filter, then* $\mathrm{Sp}_{dp}(U) = \mathrm{Sp}_T(U) = \{\lambda_0, ..., \lambda_n\}$.

[2]Formally, $\mathbb{D} \equiv_T (\mathcal{B}, \leq_D \restriction \mathcal{B})$.

Proof. Let $\mathcal{B}$ be a base for U such that $(\mathcal{B}, \supseteq^*) \simeq \lambda_0 \times \lambda_1 \times ... \times \lambda_n$. Then

$$U \equiv_T \lambda_0 \times \lambda_1 \times ... \times \lambda_n,$$

and by Fact 5.2, $\mathrm{Sp}_T(U) = \{\lambda_0, ...\lambda_n\}$. By Remark 5.8 and Proposition 5.7, $\mathrm{Sp}_{dp}(\lambda_0 \times ... \times \lambda_n) \subseteq \mathrm{Sp}_{dp}(U, \supseteq^*)$. Finally note that for every $0 \leq i \leq n$, $\lambda_i \in \mathrm{Sp}_{dp}(\lambda_0 \times ... \times \lambda_n)$ as witnessed by the sequence

$$\langle (0, 0, ..., \underbrace{\alpha}_{i^{\text{th}}\text{place}}, ...0, 0) \mid \alpha < \lambda_i \rangle.$$

Putting all of the above together, we have

$$\begin{aligned} \{\lambda_0, \lambda_1, ..., \lambda_n\} &\subseteq \mathrm{Sp}_{dp}(\lambda_0 \times ... \times \lambda_n) \\ &\subseteq \mathrm{Sp}_{dp}(U, \supseteq^*) \\ &\subseteq \mathrm{Sp}_T(U, \supseteq^*) = \{\lambda_0, \lambda_1, ..., \lambda_n\} \end{aligned}$$

□

Corollary 5.11.

(1) *Let $\omega < \lambda_0 < \lambda_1$ be regular cardinals. It is consistent that there is an ultrafilter on ω, such that $\mathrm{Sp}_{dp}(U) = \mathrm{Sp}_T(U) = \{\lambda_0, \lambda_1\}$.*

(2) *Relative to a supercompact cardinal, it is consistent that there is supercompact cardinal κ such that for any regular cardinals $\lambda_0, \lambda_1, ..., \lambda_n > \kappa$, there is a κ-directed closed κ^+-cc forcing extension where:*
   - (a) *There is a normal ultrafilter U such that $\mathrm{Sp}_{dp}(U) = \mathrm{Sp}_T(U) = \{\lambda_0, \lambda_1, ...\lambda_n\}$.*
   - (b) $\mathrm{Sp}_{dp}(Cub_\kappa) = \mathrm{Sp}_T(Cub_\kappa) = \{\lambda_0, \lambda_1..., \lambda_n\}$.

Proof. $(1), (2a), (2b)$ follows from Proposition 5.10 and Corollaries 3.12,4.7, and 4.9 respectively. □

Gitik notified us that he was able to obtain $(2b)$ independently for the depth spectrum of the club filter at a measurable cardinal from optimal assumptions. Next, let us consider the Tukey-related notion of cohesiveness due to Kanamori [31]. This notion is also known as Galvin's property [6, 9, 7, 2]:

Definition 5.12. An ultrafilter U is (μ, λ)-*cohesive* if for any $\{X_\alpha \mid \alpha < \lambda\} \in [U]^\lambda$ there is $I \in [\lambda]^\mu$ such that $\bigcap_{i \in I} X_i \in U$.

Thus, $\lambda \in \mathrm{Sp}_T(U)$ if and only if U is *not* (λ, λ)-cohesive (see [4]). The ultrafilters constructed in this paper can also be used to separate the notion of (λ_0, λ_1)-cohesive from being (λ_0, λ_0) and (λ_1, λ_1)-cohesive. It is easy to see that any non (λ_0, λ_1)-cohesive ultrafilter is not (λ_0, λ_0) and not (λ_1, λ_1)-cohesive. The converse is not true in general, as witnessed by our ultrafilters:

Theorem 5.13. *If U is a simple $P_{\lambda_0 \times \lambda_1}$-point over κ for regular cardinals $\kappa < \lambda_0 < \lambda_1$, then U is not (λ_0, λ_0) and (λ_1, λ_1)-cohesive but it is (λ_0, λ_1)-cohesive.*

Proof. As mentioned in the paragraph following Definition 5.12, not being (λ, λ)-cohesive is equivalent to $\lambda \in \mathrm{Sp}_T(U)$. Hence the first part follows from Proposition 5.10. For the second part, let us prove that U is (λ_0, λ_1)-cohesive. Fix $\langle X_i \mid i < \lambda_1 \rangle \subseteq U$. We need to find λ_0-many sets whose intersection is in U. Let $\mathcal{B} = (b_{i,j})_{(i,j)\in\lambda_0\times\lambda_1}$ be a base for U witnessing that U is a simple $P_{\lambda_0\times\lambda_1}$-point. For each $i < \lambda_1$ there is $\beta_i < \lambda_1$ and $\alpha_i < \lambda_0$ such that $b_{\alpha_i,\beta_i} \subseteq^* X_i$. There is $I \in [\lambda_1]^{\lambda_1}$ and $\alpha^* < \lambda_0$ such that for every $i \in I$, $\alpha_i = \alpha^*$. Consider the first λ_0-many indices $\{\beta_{i_\gamma} \mid \gamma < \lambda_0\} \subseteq I$. Let $\beta^* = \sup_{\gamma<\lambda_0} \beta_{i_\gamma} + 1 < \lambda_1$. Then $b_{\alpha^*,\beta^*} \subseteq^* b_{\alpha^*,i_\gamma} \subseteq^* X_{i_\gamma}$ for every $\gamma < \lambda_0$. Since $\lambda_0 > \kappa$ is regular, we can find $J \subseteq \{i_\gamma \mid \gamma < \lambda_0\}$ still of size λ_0 and some $\xi < \kappa$ such that $b_{\alpha^*,\beta^*} \setminus \xi \subseteq \bigcap_{j\in J} X_j$, as wanted. □

As in the case of the linear iteration of the Mathias forcing, our model can exhibit a small ultrafilter number.

Proposition 5.14. *Relative to a supercompact cardinal, there is a supercompact cardinal κ such that for any regular cardinals*

$$\kappa < \lambda_0 < \lambda_1 < \ldots < \lambda_n < \lambda_{n+1},$$

there is a κ-directed closed, κ^+-cc generic extension with a simple $P_{\lambda_0\times\ldots\times\lambda_n}$-point ultrafilter and $2^\kappa = \lambda_{n+1}$.

Proof. Let S, $\{X_a \mid a \in \lambda_0 \times \ldots \times \lambda_{n-1} \times S\}$, and U be as in the proof of Theorem 4.2, where $S \subseteq \lambda^+_{n+1}$ is a stationary set, and note that we skipped λ_n. By the items before Claim 4.3, and by shrinking S if necessary, we may assume that the limit s^* of the first $\lambda_{n+1} + \lambda_n$-many points of S satisfies that:

$$U^* = U \cap V[G_{\prod_{i<n}\lambda_i\times s^*}] \in V[G_{\prod_{i<n}\lambda_i\times s^*}] = V^*$$

We claim that U^* and V^* are as wanted. Clearly, in V^*, $2^\kappa = \lambda_{n+1}$. Also U^* is an V^*-ultrafilter on κ, which is generated by $\{X_a \mid a \in \lambda_0 \times \ldots \times \lambda_{n-1} \times (S \cap s^*)\}$ hence U^* is a simple $P_{\lambda_0\times\ldots\times\lambda_{n-1}\times(S\cap s^*)}$-point. Since $\lambda_0 \times \ldots \times \lambda_{n-1} \times \lambda_n$ cofinally embeds into $\lambda_0 \times \ldots \times \lambda_{n-1} \times S \cap s^*$, by Proposition 2.1, U^* is a simple $P_{\lambda_0\times\ldots\times\lambda_{n-1}\times\lambda_n}$-point. □

In the following theorem we use the well-known characterization of the generalized bounding and dominating numbers using the club filter (see for example [10]):

Proposition 5.15. *Let κ be a regular cardinal. Then*

$$\mathfrak{d}_\kappa = \min\{|\mathcal{B}| \mid \mathcal{B} \subseteq^*\text{-generates } Cub_\kappa\}$$

$$\mathfrak{b}_\kappa = \min\{|\mathcal{B}| \mid \mathcal{B} \text{ is unbounded in } (Cub_\kappa, \supseteq^*)\}$$

Note that being unbounded in $(\mathrm{Cub}_\kappa, \supseteq^*)$ is equivalent to not having a pseudo-intersection (which is just unbounded in κ).

Theorem 5.16. *Relative to a supercompact cardinal, it is consistent that κ is a supercompact cardinal and $\mathfrak{b}_\kappa < \mathfrak{d}_\kappa = \mathfrak{u}_\kappa < 2^\kappa$*

Proof. Let V^* be the model of Proposition 5.14 where $2^\kappa = \lambda_2 > \lambda_1 > \lambda_0$ and there is a simple $P_{\lambda_0\times\lambda_1}$-point over κ. By Corollary 4.9 the club filter is a simple $P_{\lambda_0\times\lambda_1}$-point and in particular generated by λ_1-many sets. By the previous proposition, in V^*, $\mathfrak{d}_\kappa = \lambda_1$. Also, any simple $P_{\lambda_0\times\lambda_1}$-point has an unbounded family of size λ_0 and therefore by the previous proposition $\mathfrak{b}_\kappa = \lambda_0$. In ZFC, for uncountable cardinals we have the following inequalities (see for example [10]):

$$\mathfrak{d}_\kappa \leq \mathfrak{u}_\kappa \leq \mathfrak{u}^{com}_\kappa.$$

Since U^* is also generated by λ_1-many sets, we have $\mathfrak{u}^{com}_\kappa \leq \lambda_1$ and we get that $\mathfrak{d}_\kappa = \mathfrak{u}_\kappa = \mathfrak{u}^{com}_\kappa = \lambda_1$. □

Let us conclude this paper with two remarks:

Remark 5.17. The above model is essentially different from the linear Mathias iteration in the sense that a P_λ-point cannot exist in those models: indeed, if there is a simple P_λ-point then $\mathfrak{b}_\kappa = \mathfrak{d}_\kappa$ (see [10]), which is not going to hold here.

Remark 5.18. The existence of a simple $P_{\mathbb{D}}$-point ultrafilter U poses restrictions on cardinal characteristics. For example, by [10] on a measurable cardinal κ, $\min(\mathrm{Sp}_T(\mathbb{D})) = \min(\mathrm{Sp}_T(U)) \leq \mathfrak{b}_\kappa$ and $\mathfrak{ch}(\mathbb{D}) = \mathfrak{ch}(U) \geq \mathfrak{d}_\kappa$. However, it is unclear if this equality must hold or if there are any other limitations (See Questions 6.7-6.9).

6. Problems

Question 6.1. It is consistent to have a simple $P_{\lambda_0\times\lambda_1\times\lambda_2}$-point ultrafilter on ω for some regular cardinals $\omega < \lambda_0 < \lambda_1 < \lambda_2$?

More generally we ask:

Question 6.2. What is the class of countably directed well-founded partial orders $\mathbb{D}$ that can be realized as a simple $P_{\mathbb{D}}$-point ultrafilter over ω? over a measurable cardinal?

Question 6.3. What kind of ill-founded directed sets $\mathbb{D}$ can be realized as simple $P_{\mathbb{D}}$-point ultrafilters?

Question 6.4. Is there a method to realize a poset $\mathbb{D}$ as a simple $P_{\mathbb{D}}$-point ultrafilter for $\mathbb{D}$'s which are not countably-directed?

Some limitations must be placed as by Corollary 2.4 no simple $P_{\omega\times\omega_1}$-point exists. But what are the exact limitations?

In this paper, we focused on $\subseteq^*$. Still, one might also be interested in the order $\supseteq$ on an ultrafilter U, especially since it is connected to the Tukey-type of $(U, \supseteq)$.

Question 6.5. What are the possible isomorphism types of $(\mathcal{B}, \supseteq)$, where $\mathcal{B}$ is a $\subseteq$-generating set of an ultrafilter?

Question 6.6. Is it consistent that $\mathfrak{d}_\kappa < \mathfrak{u}_\kappa < 2^\kappa$ for a measurable cardinal κ?

Finally, we would like to ask about the possible limitations that the existence of a simple $P_{\mathbb{D}}$-point poses.

Question 6.7. Let κ be a measurable cardinal. Can there be κ-complete simple $P_{\mathbb{D}_0}$-point and a simple $P_{\mathbb{D}_1}$-point ultrafilters on κ for two posets $\mathbb{D}_0, \mathbb{D}_1$ such that $\mathbb{D}_0 \not\equiv_T \mathbb{D}_1$?

For linear orders $\mathbb{D}_0, \mathbb{D}_1$ this was proven to be impossible in [10].

Question 6.8. Let κ be a measurable cardinal and suppose that there is a κ-complete simple $P_{\mathbb{D}}$-point ultrafilter on κ. Is any of the following statements provable:

(1) $\mathfrak{b}_\kappa = \min(\mathrm{Sp}_T(\mathbb{D}))$.
(2) $\mathfrak{d}_\kappa = \mathfrak{d}(\mathbb{D})$. Here $\mathfrak{d}(\mathbb{D})$ denotes the minimal size of a cofinal subset of $\mathbb{D}$.
(3) $\mathrm{Sp}_T(\mathbb{D}) = \mathrm{Sp}_T(\mathrm{Cub}_\kappa)$.

Answering the question for a specific non-linear $\mathbb{D}$ (e.g. $\mathbb{D} = \lambda_0 \times \lambda_1$) would also be of interest.

On ω, [14] showed that it is consistent to have a simple $P_{\aleph_1}$-point and a simple $P_{\aleph_2}$-point. Nykos [37] proved that if there is a simple P_λ-point on ω then either $\lambda = \mathfrak{b}_\kappa$ or $\lambda = \mathfrak{d}_\kappa$. The following questions relate to generalizations of these facts:

Question 6.9. Can there be $\mathbb{D}_0, \mathbb{D}_1, \mathbb{D}_2$ non-Tukey equivalent such that there are simple $P_{\mathbb{D}_i}$-point ultrafilters on ω for $i = 0, 1, 2$?

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