The Forcing Method and Prikry-Type Forcing

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The Forcing Method- Background and Motivation

Hilbert's first problem[**?**]:

Problem

Is $2^{\aleph_0}$ *the first uncountable cardinality? Equivalently*[a]*, is* $\aleph_1 = 2^{\aleph_0}$*?*

[a]Along this talk we assume the axiom of choice

Definition (Continuum Hypothesis [2])

For every set A, if $\aleph_0 < |A| \leq 2^{\aleph_0}$, then $|A| = 2^{\aleph_0}$. We denote this statement by CH.

The following celebrated result is due to Gödel and Cohen [7],[3],[4].

Theorem (K. Gödel, P. Cohen)

The statement CH *cannot be proved nor refuted merely from* ZFC.

CH is the first concrete evidence of a statement which is *undecidable* from ZFC i.e. $ZFC \nvdash CH \wedge ZFC \nvdash \neg CH$.

How to Prove Undecidability?

The idea to produce an undecidability proof, is based on the soundness lemma:

Lemma (Soundness Lemma for ZFC)

If ϕ is a statement in the language of set theory and $ZFC \vdash \phi$, then in every model of ZFC, ϕ holds.

Corollary

If there is a model of ZFC such that $\neg\phi$ holds, then $ZFC \nvdash \phi$

Thus, if we would like to prove the undecidability of CH from ZFC, we can simply find two models of ZFC, one in which CH holds and one in which $\neg CH$ holds.

Simple Analogy in Group Theory

Let us consider the three axioms of a group Gr in the language $\{e, *\}$:

1. $\forall x.\forall y.\forall z.x * (y * z) = (x * y) * z$ (Associativity).
2. $\forall x.e * x = x * e = x$ (Identity element).
3. $\forall x.\exists y.x * y = y * x = e$ (Inverse element).

Moreover, let us consider the abelian group statement "$\forall x.\forall y.x * y = y * x$" which we denote by AB. we would like to prove that AB is undecidable from the axioms Gr.
The first direction, that $Gr \nvdash AB$, so we can simply find a group that satisfy $\neg AB$, for example S_3. In this analogy, S_3 is the model which we use to conclude that $Gr \nvdash AB$.
The second direction, that $Gr \nvdash \neg AB$, we consider the model $\langle\{0\}, +\rangle$ as an example for a model in which AB holds.
So we conclude that AB is undecidable from the axioms Gr. The prove for the undecidabily of CH from ZFC is similar, the difficulty however is to produce models for ZFC...

The constructible universe L

Gödel first produced a model of ZFC which satisfied CH (and many more important statements), this model is called **the constructible universe**, and is denoted by L.
Roughly speaking, we define an operation Def, Such that for every set A, $Def(A) \subseteq P(A)$, and $Def(A)$ is the collection of all definable subsets of A. Then L is obtained by transfinite recursion of this operation starting from $\emptyset$.
Gödel proved that L is a model of ZFC which satisfy CH, thus established the proof of the theorem:

Theorem (K. Gödel)

$ZFC \nvdash \neg CH$.

Note that at this point it is still possible that $ZFC \vdash CH$.

Forcing- A Method to Produce Models of Set Theory

The second model was produced in the 60's by Paul Cohen who invented a method called forcing, which produces *ZFC* models. He was able to force a model of *ZFC* in which $\neg CH$ holds.

Theorem (P. Cohen)

ZFC $\nvdash$ *CH*.

Since Cohen original usage, forcing as been one of the key tools for many undecidability proofs of statement in various areas of mathematics. Today it is one of the must important fields of set theory.
In what come next we will review some basic theory of forcing and try to give some intuition of how the mechanism of this method works.

What is Forcing?

We start with any transitive model of ZFC usually we denote this by V–**The ground model** over which we force and produce a different model (transitive means that for every $x \in V$ and $y \in x$, $y \in V$. If $y \subseteq x$ then it does not necessarily follow that $y \in V$).

Definition

A forcing notion is a poset $\langle \mathbb{P}, \preceq \rangle$, which belongs to V, with a smallest element denoted by $0_{\mathbb{P}}$.

There are many forcing notions, the following one is the original forcing Cohen used:

Example

Cohen forcing is defined as

$$Cohen(\omega) = \{f \subseteq \omega \times \{0,1\} \mid f \text{ is a partial function and } |f| < \aleph_0\}$$

The order is $f \leq g \leftrightarrow f \subseteq g$. It is the set of finite approximation of a function f^* from ω to $\{0,1\}$.

Generic Filter

To generate the new model, we add to V a new object– a **Generic filter over** V and close under set theoretical operations.

Definition (Generic filter)

Let $\mathbb{P} \in V$ be a forcing notion. A set $G \subseteq \mathbb{P}$ is a V-generic filter if:

1. $\forall p \in G. \forall q \leq p. q \in G$. ($G$ is downward closed).
2. $\forall p_1, p_2 \in G. \exists r \in G. p_1, p_2 \leq r$. (every two elements of G have a common extension in V, this requirement is similar to a the requirement of a directed system)
3. For every dense subset $D \subseteq \mathbb{P}$, $D \in V$, $D \cap G \neq \emptyset$. Where dense means that for every $\forall p \in \mathbb{P}. \exists d \in D. p \leq d$.

The model obtained by adding G to V is denoted by $V[G]$.

Forcing Illustration

Recall

$$Cohen(\omega) = \{f \subseteq \omega \times \{0,1\} \mid f \text{ is a partial function and } |f| < \aleph_0\}$$

with the order $f \leq g \leftrightarrow f \subseteq g$.
Let $G \subseteq Cohen(\omega)$ be a generic filter over V, then G is a set of partial function in $\{0,1\}^\omega$. By (2) of the definition of generic, $f^* = \cup G = \cup_{f \in G} f$ is also a partial function in $\{0,1\}^\omega$. Condition (3) ensures for that:

Claim

f^* is a full function on ω. And $f^* \in V[G] \setminus V$.

Proof.

Let $n < \omega$, define $D_n = \{g \in \mathbb{P} \mid n \in Dom(g)\}$. It is not hard to check that D_n is dense. By (3), there is $g \in D_n \cap G$, thus $n \in Dom(g) \subseteq Dom(f^*)$. Let $h : \omega \to \{0,1\}$ be in V, we claim that $h \neq f^*$. Indeed, $D_h = \{g \in \mathbb{P} \mid \exists n \in Dom(g).g(n) \neq h(n)\}$ is dense, hence there is $g \in D_h \cap G \neq \emptyset$, and since $f^* \restriction Dom(g) = g$, it follows that for some n, $f^*(n) \neq h(n)$. □

The Fundamental Theorem of Forcing

We state the fundamental theorem of forcing to conclude this discussion:

Theorem

Let V be a transitive model of ZFC and $\langle \mathbb{P}, \leq_{\mathbb{P}} \rangle \in V$ a forcing notion. Let G be a generic filter over V for the forcing $\mathbb{P}$, then there is a model $V[G]$ such that:

1. *$V[G]$ is a transitive model of ZFC.*
2. *$V \subseteq V[G]$ and $G \in V[G]$.*
3. *$V[G]$ is the minimal ZFC model containing V and G.*
4. *V and $V[G]$ have the same ordinals*

Although V and $V[G]$ have the same ordinals, they do not necessarily have the same cardinals.

Cardinals

Quick reminder of the basic definitions of cardinals

Definition (Cardinal- Reminder)

An ordinal α is a cardinal if for every $\beta < \alpha$, $|\beta| < |\alpha|$. Equivalently, for every $\beta < \alpha$, there is no onto function $f : \beta \to \alpha$. For every ordinal β, $|\beta|$ denotes the unique cardinal α such that there is a bijection between β and α.

Definition (The $\aleph_\alpha$ construction)

For an ordinal β, $(\beta)^+$ is the first cardinal α, such that $\beta < \alpha$.
We define recursively: $\aleph_1 = \aleph_0^+$, $\aleph_2 = \aleph_1^+$,... $\aleph_{\alpha+1} = \aleph_\alpha^+$. At limit ordinals, δ, we define $\aleph_\delta = \sup_{\alpha<\delta}\aleph_\alpha$.

Every $\aleph_\alpha$ is a cardinal and every cardinal is some $\aleph_\alpha$.
What determine the cardinals of each transitive *ZFC* models is the functions in that model. For example, $\aleph_1$ in the sense of a model V (denoted by $\aleph_1^V$) is the first ordinal α such that there is no function $f \in V(!)$ from ω onto α.

Preservation of Cardinals

After forcing, we obtain two transitive models of ZFC, $V \subseteq V[G]$.

Claim

If α is a cardinal in $V[G]$ then α is a cardinal in V. In other words $Car^{V[G]} \subseteq Car^{V}$.

If α is cardinal in V, it might seize to be cardinal in $V[G]$, since a witnessing function $f : \beta \to \alpha$ onto from some $\beta < \alpha$ can exists in $V[G] \setminus V$.

Example (Levi Collapse[?])

Define $Col(\aleph_0, \aleph_1) = \{f \subseteq \omega \times \omega_1 \mid f \text{ is partial function and } |f| < \aleph_0\}$, $f \leq g \leftrightarrow f \subseteq g$.

Claim

Let G be a V-generic filter for $Col(\aleph_0, \aleph_1)$, then $V[G] \models \aleph_1^V$ is not a cardinal, and therefore, $\aleph_1^V < \aleph_1^{v[G]}$.

Preservation of Cardinals

Proof.

For every $\alpha < \omega_1$, the set $D_\alpha = \{g \in Col(\aleph_0, \aleph_1) \mid \alpha \in Im(g)\}$ is dense. Hence, the function $f_G = \cup G : \omega \to \omega_1^V$ is onto. This means that in $V[G]$, $\aleph_1^V$ is not a cardinal, also every $\omega < \beta < \omega_1^V$ is not a cardinal, hence by minimality $\aleph_1^V < \aleph_1^{V[G]}$. □

One of Cohen's main lemmas regarding Cohen forcing is the following:

Lemma

$Cohen(\omega)$ preserves cardinals. i.e. if G is a generic filter for $Cohen(\omega)$ then V and $V[G]$ have the same cardinals.

This was a crucial step in the proof that CH fails in some forcing extension $V[G]$. For example, if we aim to find a forcing (in V) which adds $\aleph_2$ many functions in $\{0,1\}^\omega$ to $V[G]$, then it is $\aleph_2^V$, so all we get is $V[G] \models \aleph_2^V \leq 2^{\aleph_0}$. If the forcing collapses $\aleph_2^V$, then in $V[G]$, CH can still be true since $\aleph_2^V < \aleph_2^{V[G]}$. However, if the forcing preserves cardinal, then in $V[G]$ the continuum will be at least $\aleph_2^{V[G]}$, and CH fails.

Preservation of Cardinals

Definition (cofinality of ordinals)

Let κ be an ordinal, the cofinality of κ, denoted by $cof(\kappa)$ is the least γ such that there is a sequence of length γ, $\langle \alpha_i \mid i < \gamma \rangle$ such that $\alpha_i < \kappa$ and $\sup_{i<\gamma}(\alpha_i) = \kappa$.

Every successor cardinal κ is regular i.e. it satisfy $cof(\kappa) = \kappa$.
Similar to cardinals, note that the cofinality of an ordinal can vary between models.

Theorem

Let $\mathbb{P}$ be a cofinalities preserving forcing i.e. for every V-generic $G \subseteq \mathbb{P}$ and every ordinal α, $cof(\alpha)^V = cof(\alpha)^{V[G]}$. Then $\mathbb{P}$ is a cardinals preserving forcing.

Proof.

Assume that in $V[G]$ all the cofinalities are preserved, and let α be the first such that $\aleph_\alpha^V < \aleph_\alpha^{V[G]}$. Therefore in $V[G]$, $\aleph_\alpha^V$ is not a cardinal, hence $(Cof(\aleph_\alpha^V))^{V[G]} < \aleph_\alpha^V$. On the other hand, $\aleph_\alpha^V$ must be a successor cardinal in V. Hence $(Cof(\aleph_\alpha^V))^V = \aleph_\alpha^V$. This is a contradiction, since $(Cof(\aleph_\alpha^V))^V = (Cof(\aleph_\alpha^V))^{V[G]}$. □

The Other Direction

A natural question rises:

Question

Is every cardinal preserving forcing necessarily preserves cofinalities?

In order to construct a counter example [9], Karel Prikry had to assume additional assumption regarding the set theoretical universe – the existence of a large cardinal.

Definition (Ultrafilter)

A κ-complete ultrafilter on a cardinal κ, is a set $U \subseteq P(\kappa)$ such that:

1. For every $X \subseteq \kappa$, either $X \in U$ or $\kappa \setminus X \in U$. Moreover, $\emptyset \notin U$ and $\kappa \in U$.
2. $\forall X \in U. \forall Y \supseteq X. Y \in U$.
3. The intersection of less than κ many sets in U is in U i.e. if $\langle A_i \mid i < \lambda \rangle$, $A_i \in U$ and $\lambda < \kappa$, then $\cap_{i<\lambda} A_i \in U$ (generalization the σ-completness requirement in a σ-algebra).

Example

Fix $\alpha < \kappa$ and set $U_\alpha = \{X \subseteq \kappa \mid \alpha \in X\}$. It is a κ-complete ultrafilter over κ. This kind of ultrefilters is called trivial.

The idea behind ultrafilter is to have some notion of a "large set", one should think of measure one sets with respect to some probability function. The major difference is that for ultrafilter, every set is measurable.

Definition (Measurable cardinal)

A measurable cardinal is a cardinal which carries a κ-complete non trivial ultrafilter.

It turns out to be a very strong assumption:

Theorem

ZFC $\nvdash \exists$ *a measurable cardinal*

A measurable cardinal is a type of large cardinal. It is known that a measurable cardinal is regular i.e. $cof(\kappa) = \kappa$.

Prikry Forcing

Definition (Prikry forcing)

Let κ be a measurable cardinal and U a (normal) κ-complete ultrafilter over κ. Prikry forcing, denoted by $\mathbb{P}(U)$ is the set of all finite sequence $\langle \alpha_1, ..., \alpha_n, A \rangle$, where

1. $\alpha_1 < ... < \alpha_n < \kappa$.
2. $A \in U$ and $\min(A) > \alpha_n$.

The order is defined: $\langle \alpha_1, ..., \alpha_n, A \rangle \leq \langle \beta_1, ..., \beta_m, B \rangle$ iff

1. $n \leq m$.
2. $\langle \alpha_1, ..., \alpha_n \rangle = \langle \beta_1, .., \beta_n \rangle$.
3. For every $n < i \leq m$, $\beta_i \in A$.
4. $B \subseteq A$.

The idea is that each condition is a finite approximation of a unbounded ω-sequence in a measurable cardinal κ, the measure one set A is a set of candidates for the continuation of the sequence.

Prikry Forcing Properties

If G is generic for $\mathbb{P}(U)$, then $C_G = \{\alpha < \kappa \mid \alpha$ apears in some condition $p \in G\}$ is a set of order type ω unbounded in κ, definable in $V[G]$. Hence $cof^{V[G]}(\kappa) = \omega$.

Theorem (Prikry forcing generic extension)

1. *Let G be a generic filter of $\mathbb{P}(U)$, then $cof^{V[G]}(\kappa) = \omega < \kappa = cof^{V}(\kappa)$.*
2. *$\mathbb{P}(U)$ preserves cardinals.*

Measurablity is crucial in order to preserve cardinals.[5]
Although κ is no longer a measurable cardinal in $V[G]$, some properties still hold in $V[G]$. Having said that, it is no surprise that Prikry forcing plays a key role in the modern analysis of singular cardinal arithmetic and produce models in which cardinal arithmetic has extreme behavior.
To obtain models in which the cofinality is changed to some uncountable cardinal, Menachm Magidor introduced Magidor forcing [8], denote by $\mathbb{M}[\vec{U}]$. It has similar features to Prikry forcing, it changes the cofinality of a measurable cardinal while preserving all cardinals.

Our Work

The intermediate models of Prikry forcing extensions are classified by the following theorem[6]:

Theorem (Gitik, Kanovei and Koepke)

If $G \subseteq \mathbb{P}(U)$ is a V-generic filter, and $V \subseteq M \subseteq V[G]$ is a transitive ZFC model, then there is a V-generic filter $H \subseteq \mathbb{P}(U)$ such that $M = V[H]$.

Our generalization is[1]:

Theorem (B. and Gitik)

If $G \subseteq \mathbb{M}[\vec{U}]$ is a V-generic filter, and $V \subseteq M \subseteq V[G]$ is a transitive ZFC model, then there is a V-generic filter H for a "Magidor-Like" forcing $\mathbb{Q}$ such that $M = V[H]$. Where "Magidor-like" refers to a class of forcing which is a generalization of $\mathbb{M}[\vec{U}]$.

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Finish Line

Thank you for your attention!