Homework 7

MATH 215 (due October 21) October 14, 2022

Problem 1. Compute that following sets, no proof required:

1. $\{\emptyset, \{\emptyset\}\} \times \{\emptyset, 1\}$.

2. a_4, where a_n is defined recursively by

$$a_0 = 1, a_1 = 1, \text{ and}$$

$$a_{n+1} = a_n - 2a_{n-1}$$

3. A_3, where A_n is defined recursively by

$$A_0 = \emptyset, \text{ and}$$

$$A_{n+1} = A_n \cup \{A_n\}.$$

Problem 2. Prove that for every sets A, B, C,

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

Problem 3. A *geometric sequence* is a sequence a_n with a constant successive ratio, namely

$$\exists q . \forall n \in \mathbb{N} . \frac{a_{n+1}}{a_n} = q$$

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1. Prove, by definition, that the sequence $a_n = 5 \cdot 2^n$ is geometric.

2. Prove the following closed formula for a geometric sequence a_n with constant ratio q:

$$\forall n \in \mathbb{N} . a_n = a_0 \cdot q^n$$

3. Prove the formula for the sum of a geometric sequence a_n with constant ratio q:

$$\forall N \in \mathbb{N} . a_0 + a_1 + a_2 + \ldots + a_N = a_0 \frac{1 - q^{N+1}}{1 - q}$$

.

4. Compute the sum

$$5 + 5^2 + \ldots. + 5^{100}$$

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Problem 4 (optional)**.** Let a_n be a sequence of positive numbers. Define the sequence $b_n = \log(a_n)$. Prove that:

$$a_n \text{ is geometric } \Leftrightarrow\ b_n \text{ is arithmetic}$$

[Hint: use the identity $\log(a \cdot b) = \log(a) + \log(b)$]

Problem 5. Define recursively the sequence A_n by

$$A_0 = \{1, 2, 3\}$$

$$A_{n+1} = \{a \cdot b \mid a, b \in A_n\}$$

1. Compute A_2. No proof required.

2. Prove that for every $n \in \mathbb{N}$ $A_n \subseteq A_{n+1}$.

3. Prove that for every $n \in \mathbb{N}$, $2^{n+1} \in A_n$.

Problem 6 (Optional)**.** Define the sum of pairs by

$$\langle a, b\rangle + \langle c, d\rangle = \langle a + c, b + d\rangle.$$

Define the multiplication of a real number t with a pair of real numbers $\langle a, b\rangle$ by

$$t \cdot \langle a, b\rangle = \langle t \cdot a, t \cdot b\rangle$$

1. Compute $\langle 1, 2\rangle + 2 \cdot \langle -1, 2\rangle$.
2. Prove that if $\langle a_0, b_0\rangle \in \mathbb{R}^2$ is a pair such that for every pair $\langle a, b\rangle \in \mathbb{R}^2$ $\langle a_0, b_0\rangle + \langle a, b\rangle = \langle a, b\rangle$, then $\langle a_0, b_0\rangle = \langle 0, 0\rangle$.
3. Prove that fore every $t \in \mathbb{R}$ and $\langle a, b\rangle \in \mathbb{R}^2$:

$$t \cdot \langle a, b\rangle = \langle 0, 0\rangle \Leftrightarrow (t = 0 \vee \langle a, b\rangle = \langle 0, 0\rangle)$$