MATH 300 **Homework 7-Sols**

Problem 1. 1. $f_1 : \mathbb{R} \to codom(f_1)$, defined by $f_1(x) = \{x^2\}$.

Compute $f_1(5)$.

2. $f_2 : P(\mathbb{N}) \to codom(f_2)$, defined by $f_2(x) = \begin{cases} \min(x) & 4 \in x \\ x & else \end{cases}$.

Compute $f_2(\mathbb{N}_{even})$ and $f_2(\{n \in \mathbb{N} \mid n^2 - 2n + 1 \leq 9\})$.

3. $f_3 : \mathbb{N} \times \mathbb{Z} \to codom(f_3)$, defined by $f_3(\langle n, m \rangle) = \{x \in \mathbb{N} \mid n < x < m\}$.

Compute $f_3(\langle 1, 5 \rangle)$ and $f_3(\langle 1, -1 \rangle)$.

Solution.

1. $f_1(5) = \{5^2\} = \{25\}$

2. $f_2(\mathbb{N}_{even}) = 0$

   $f_2(\{n \in \mathbb{N} \mid n^2 - 2n + 1 \leq 9\}) = 0$

3. $f_3(\langle 1, 5 \rangle) = \{2, 3, 4\}$

   $f_3(\langle 1, -1 \rangle) = \emptyset$

Problem 2. For each of the functions from the previous exercise, find their domain and codomain.

Solution.

1. $dom(f_1) = \mathbb{R}$, $codom(f_1) = P(\mathbb{R})$
2. $dom(f_2) = P(\mathbb{N})$, $codom(f_2) = \mathbb{N} \cup P(\mathbb{N})$
3. $dom(f_3) = \mathbb{N} \times \mathbb{Z}$, $codom(f_3) = P(\mathbb{N})$

Problem 3. Define

$$f_1 : \mathbb{N} \to \mathbb{N} \times \mathbb{N}, \quad f_1(n) = \langle n+1, n+2 \rangle$$

$$f_2 : \mathbb{N} \to \mathbb{N}, \quad f_2(n) = n^2$$

$$f_3 : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}, \quad f_3(\langle n, m \rangle) = n - m$$

$$f_4 : \mathbb{N} \to \mathbb{N}, \quad f_4(n) = n + 1$$

Determine if the following compositions are defined and compute them:

1. $f_1 \circ f_2$ and $f_2 \circ f_1$.
2. $f_2 \circ f_2$. and $f_3 \circ f_3$
3. $f_4 \circ f_2$ and $f_2 \circ f_4$.
4. $f_3 \circ f_1 \circ f_2$ and $f_4 \circ f_3 \circ f_2$.

Solution.

1. $f_1 \circ f_2 : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, $(f_1 \circ f_2)(n) =< n^2 + 1, n^2 + 2 >$

   $f_2 \circ f_1$ is undefined.

2. $f_2 \circ f_2 : \mathbb{N} \to \mathbb{N}$, $(f_2 \circ f_2)(n) = n^4$

   $f_3 \circ f_3$ is undefined.

3. $f_4 \circ f_2 : \mathbb{N} \to \mathbb{N}$, $(f_4 \circ f_2)(n) = n^2 + 1$

   $f_2 \circ f_4 : \mathbb{N} \to \mathbb{N}$, $(f_2 \circ f_4)(n) = (n+1)^2$

4. $f_3 \circ f_1 \circ f_2 : \mathbb{N} \to \mathbb{Z}$, $(f_3 \circ f_1 \circ f_2)(n) = -1$

   $(f_4 \circ f_3 \circ f_2)(n)$ is undefined

Problem 4. For a function $f : A \to B$ and $C \subseteq A$ define the *pointwise image of C by f* as

$$f''C = \{f(c) \mid c \in C\}$$

(a) Prove that if $f : A \to B$ is a function and $C \subseteq A$, then

$$(f''A) \setminus (f''C) \subseteq f''[A \setminus C].$$

(b) Give an example of sets A, B and a function $f : A \to B$ and a subset $C \subseteq A$ such that

$$(f''A) \setminus (f''C) \neq f''[A \setminus C].$$

(c) Prove that if $f : A \to B$ is an injection and $C \subseteq A$, then

$$(f''A) \setminus (f''C) = f''[A \setminus C].$$

Solution.

(a) Let $b \in f''A \setminus f''C$. Since $b \in f''A$, there is $a \in A$ such that $b = f(a)$. Since $b \notin f''C$, $a \notin C$. It follows that $a \in A \setminus C$. We conclude that $b = f(a) \in f''[A \setminus C]$.

(b) Let $f : \{1,2\} \to \{1,2\}$ defined by $f(1) = f(2) = 1$. Let $A = \{1,2\}$, and $C = \{1\}$. Then

$$f''\{1,2\} = \{1\},\ f''\{1\} = \{1\} \Rightarrow f''\{1,2\} \setminus f''\{1\} = \emptyset$$

Also

$$\{1,2\} \setminus \{1\} = \{2\} \Rightarrow f''[\{1,2\} \setminus \{1\}] = \{1\}$$

Hence

$$f''\{1,2\} \setminus f''\{1\} = \emptyset \neq \{1\} = f''[\{1,2\} \setminus \{1\}].$$

(c) Suppose that f is injective and we would like to prove that

$$(f''A) \setminus (f''C) = f''[A \setminus C].$$

By a double inclusion. In section (a) we proved $\subseteq$. For the other direction, let $x \in f''[A \setminus C]$. Then there is $a \in A \setminus C$ such that $f(a) = x$. By the definition of difference, we would like to prove that $x \in f''A$ and $x \notin f''C$. Since $a \in A$, it follows that $x = f(a) \in f''A$. Suppose towards a contradiction that there is $c \in C$ such that $f(c) = x$. Then $f(c) = f(a)$. Since f is injective, $c = a$. However $c \in C$ and $a \notin C$, contradiction. Hence $x \in f''C$.