Homework 4-Sols

MATH 300 (due Feb 23) Feb 16, 2022

Problem 1. (1) Prove that $\lim_{n\to\infty} \frac{n^2+2}{2n^2+1} = \frac{1}{2}$.

Solution WTP $\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \geq N$, $|\frac{n^2+2}{2n^2+1} - \frac{1}{2}| < \epsilon$.

Let $\epsilon > 0$,(side computation: $\frac{n^2+2}{2n^2+1} - \frac{1}{2} = \frac{2n^2+4-2n^2-1}{4n^2+2} = \frac{3}{4n^2+2} < \epsilon$ $\frac{3}{\epsilon} < 4n^2 + 2 \Rightarrow \sqrt{\frac{\frac{3}{\epsilon}-2}{4}} < n$, note that $\frac{\frac{3}{\epsilon}-2}{4}$ might be negative so the square root might be problematic. To take care of this we just let $\sqrt{\max\{\frac{\frac{3}{\epsilon}-2}{4}, 0\}} + 1 = N$)

Define $\sqrt{\max\{\frac{\frac{3}{\epsilon}-2}{4}, 0\}} + 1 = N$ Let $n \geq N$ WTP $|\frac{n^2+2}{2n^2+1} - \frac{1}{2}| < \epsilon$. Indeed,

$$|\frac{n^2+2}{2n^2+1} - \frac{1}{2}| = |\frac{3}{4n^2+2}| = \frac{3}{4n^2+2} \leq \frac{3}{4N^2+2} < \frac{3}{4 \cdot \frac{\frac{3}{\epsilon}-2}{4} + 2} = \epsilon$$

(2) Write a formula that expresses $\lim_{n\to\infty} a_n \neq L$.

Solution $\exists \epsilon > 0 \forall N \in \mathbb{N} \exists n \geq N |a_n - L| \geq \epsilon$.

(3) Prove that for every $L \in \mathbb{R}$, $\lim_{n\to\infty}(-1)^n \neq L$.

Solution Let $L \in \mathbb{R}$. and let us split into cases:

(a) If $L = 1$, WTP $\exists \epsilon > 0 \forall N \in \mathbb{N} \exists n \geq N |(-1)^n - L| \geq \epsilon$. Define $\epsilon = 1$ and let $N \in \mathbb{N}$. Consider any odd number $n \geq N$ (for example $n = 2N + 1$). Then $(-1)^n = -1$ hence $|-1-1| = 2 > 1$.

(b) If $L \neq 1$, let $\epsilon = \frac{|L-1|}{2} < |L-1|$. Then

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Problem 2. (1) Prove that for every rational number $q \in \mathbb{Q}$ $q \neq 0$, $\sqrt{2} \cdot q$ is irrational.

Soltuion. Suppose towards contradiction that $\sqrt{2}q = p \in \mathbb{Q}$. Then $\sqrt{2} = \frac{p}{q}$. The ration of two rationals is rationals and therefore $\sqrt{2} \in \mathbb{Q}$, contradicting the theorem we saw in class.

(2) Prove or disprove: the sum or irrational numbers is irrational.

solution. Counterexample, $\sqrt{2} + (1 - \sqrt{2}) = 1$.

(3) Prove that $\sqrt{5}$ is irrational.

Soltuion. Like in WS4.

(4) (optional) Formulate a conjecture for the rationality and irrationality of real numbers of the form $\sqrt{n}$.

Solution For any natural number n, either $\sqrt{n}$ is an integer or irrational.

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Problem 3. Determine which of the following statements are true. Prove your answer:

1. $\{1, -1\} \in \{1, -1, \{1\}, \{-1\}\}$.

   Solution. Not true. The element $\{1, -1\}$ is not any of the element $1, -1, \{1\}, \{-1\}$.

2. $7 \in \{n \in \mathbb{N} \mid |n^2 - n - 3| \leq 5\}$.

   Solution. Not true. $|7^2 - 7 - 3| = 39 > 5$ and by the separation principle, $7 \notin \{n \in \mathbb{N} \mid |n^2 - n - 3| \leq 5\}$.

3. $1 \in \{\mathbb{N}, \mathbb{Z}, \mathbb{N}_{even}\}$.

   Solution. Not true, proof like 1.

4. $16 \in \{x \in \mathbb{N} \mid \forall y \in \mathbb{N}. y < 4 \Rightarrow y^2 + 2y < x\}$.

   Solution. True. By the separation principle, we want to prove that $16 \in \mathbb{N}$ and $\forall y \in \mathbb{N}. y < 4 \Rightarrow y^2 + 2y < 16$. Let $y \in \mathbb{N}$ and suppose that $y < 4$, then either $y = 0, 1, 2, 3$. Let us prove the universal statement one-by-one.

   (1) $y = 0$, $0^2 + 2 \cdot 0 = 0 < 16$.

   (2) $y = 1$, $1^2 + 2 = 3 < 16$.

   (3) $y = 2$, $2^2 + 4 = 8 < 16$.

   (4) $y = 3$, $3^2 + 6 = 15 < 16$.

   Therefore $16 \in \{x \in \mathbb{N} \mid \forall y \in \mathbb{N}. y < 4 \Rightarrow y^2 + 2y < x\}$.

Problem 4. Compute the following sets using the list principle and global symbols $\mathbb{N}, \mathbb{N}_{even}, \mathbb{N}_{odd}$ and $\mathbb{Z}$. No proof in needed.

1. $\{x \in \mathbb{N} \mid \exists k \in \mathbb{N}. k + x \in \mathbb{N}_{even}\}$.

   Solution. $\{x \in \mathbb{N} \mid \exists k \in \mathbb{N}. k + x \in \mathbb{N}_{even}\} = \mathbb{N}$.

2. $\{x \in \mathbb{N} \mid x^2 + 2x - 3 = 0\}$.

   Solution. $\{x \in \mathbb{N} \mid x^2 + 2x - 3 = 0\} = \{1\}$.

3. $\{x \in \mathbb{Z} \mid \forall y \in \mathbb{N}. y < x \Rightarrow y^2 < x^2\}$

   Solution. $\{x \in \mathbb{Z} \mid \forall y \in \mathbb{N}. y < x \Rightarrow y^2 < x^2\} = \mathbb{Z}$.

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Problem 5. Find a formal expression for the following sets:

1. The set of all integers below 100 which are are divisible by 3.

   Solution. $\{x \in \mathbb{Z} \mid \exists k \in \mathbb{Z}(x = 3k)\}$.

2. The set of all integers which are the successor of a power of 2.

   Solution. $\{2^n + 1 \mid n \in \mathbb{N}\}$.

3. The set of all (exactly) two element sets of real numbers.

   Solution. $\{\{a, b\} \mid a, b \in \mathbb{R}, a \neq b\}$.