Homework 3-Sols

Problem 1. Prove the following equivalences (using a double implication):

An integer is divisible by 4 if and only if its last two digits form a number divisible by 4.

[Hint: Decompose $n = 100l + d$ where k, l is some integers and $0 \leq d \leq 99$. Then the number d is the last two digits.]

Solution Let $n = 100k + d$ where $0 \leq d \leq 99$. Let us prove this equivalence using a double implication

- $\rightarrow$ Suppose that n is divisible by 4. WTP d is divisible by 4. indeed, $d = n - 100k$, and since both n and $100k$ are divisible by 4, n is divisible by 4, by a theorem we saw in class that the difference of two numbers divisible by m is divisible by m.
- $\leftarrow$ Suppose that d is divisible by 4 then $n = 100k + d$ is a sum of two numbers divisible by 4, hence divisible by 4.

Problem 2. Let $x, y \in \mathbb{R}$, prove that either $x \leq y$ or there is $n \in \mathbb{N}$ such that $xn > yn + 2025$.

Solution Assume that x, y are real numbers WTP $x \leq y$ or there is $n \in \mathbb{N}$ such that $xn > yn + 2025$. Let us split into cases:

(1) If $x \leq y$ then the "or" statement holds and we are done.

(2) If $y < x$, then $x - y > 0$ and also $\frac{x-y}{2025} > 0$. Define (we are proving an existential statement) $n = \lceil \frac{2025}{x-y} \rceil + 1$. Then $n \in \mathbb{N}$ since $\frac{2025}{x-y}$ is positive. We have that

$$\frac{1}{n} = \frac{1}{\lceil \frac{2025}{x-y} \rceil + 1} < \frac{1}{\frac{2025}{x-y}} = \frac{x-y}{2025}$$

hence

$$2025 < n(x-y) = nx - ny$$

It follows that

$$ny + 2025 < nx.$$

Problem 3. Prove that if a and b are odd integers, then $a^2 - b^2$ is a multiple of 8.

Solution Suppose that a, b are odd. WTP that $a^2 - b^2$ is divisible by 8. By assumption, there are k, l integers such that $a = 2k + 1$ and $b = 2l + 1$. It follows that

$$a^2 - b^2 = (4k^2 + 4k + 1) - (4l^2 + 4l + 1) = 4(k^2 + k - (l^2 + l))$$

In class we proved that for every n, $n^2 + n$ is even and therefore $k^2 + k - (l^2 + l)$ is the difference if two even number hence even. Hence there is an integer r such that $k^2 + k - (l^2 + l) = 2r$ and therefore

$$a^2 - b^2 = 4(k^2 + k - (l^2 + l)) = 8r$$

Therefore $a^2 - b^2$ is divisible by 8.

Problem 4. Let a, b, c be integers. Prove that if $a^2 + b^2 = c^2$, then abc is even.

Solution. Let us prove the contrapositive. WTP $a^2 + b^2 = c^2$ Suppose that abc is odd. Then a, b, c must all be odd. But then a^2, b^2, c^2. Now $a^2 + b^2$ is even as the sum of two odds. We conclude that $a^2 + b^2 \neq c^2$, since a number cannot be both even and odd.