Homework 2-Sols

Problem 1. Formalize each of the following statements using the predicate calculus.

(a) There is a number which is not divisible by any square.

Solution. $\exists x \in \mathbb{Z}(\forall y \in \mathbb{Z}((\exists k \in \mathbb{Z}, k^2 = y) \Rightarrow y \nmid x))$

(b) Every prime number is greater than 1.

Solution. $\forall p \in \mathbb{N}((\forall n \in \mathbb{N}(n|p \Rightarrow n = 1 \vee n = p)) \Rightarrow p > 1)$

Problem 2. For each of the following statements, write the negation of the sentences **without** the negation symbol "¬", and determine whether the sentence is true or false in the domain of the real numbers:

1. $\exists\epsilon((\epsilon > 0) \wedge (\forall x(x > 0 \Rightarrow x > \epsilon)))$. **Solution.**

   $$\exists\epsilon((\epsilon > 0)\wedge(\forall x(x > 0 \Rightarrow x > \epsilon))) \equiv \forall\epsilon((\epsilon \le 0)\vee(\exists x((x > 0)\wedge(x \le \epsilon)))$$

   The statement is false.

2. $\forall x((x > 5) \Leftrightarrow (\forall y(y > -100)))$.
   (Hint: Recall that $A \Leftrightarrow B \equiv (A \Rightarrow B) \wedge (B \Rightarrow A)$)

   Solution.

   $$\forall x((x > 5) \Leftrightarrow (\forall y(y > -100)) \equiv$$

   $$\equiv \exists x(((x > 5) \wedge (\exists y(y \le -100))) \vee ((x \le 5) \wedge (\forall y(y > -100)))))$$

   The statement os false.

Problem 3. Prove the following statement:

If both a and b are divisible by n, then $a + b$ is divisible by n.

Solution. Suppose that a and b are divisible by n. WTP $a+b$ is divisible by n. By assumption, there are integers k, l such that $a = kn$ and $b = ln$. Define $t = k+l$, then $nt = n(k+l) = nk+nl = a+b$. Hence $a-b$ is divisible by n. Therefore if n divides a and b then n divides $a + b$.

Problem 4. Prove the following implication:

If n is even then $n+2$ is even.

Solution. Suppose that n is even. WTP $n+2$ is even. By assumption $2|n$ and therefore there is k such that $n = 2k$. Define $t = k+1$, it follows that $2t = 2(k+1) = 2k+2 = n+2$. Hence n is even. Therefore if n is even then $n+2$ is even.