Homework 10-Sols

MATH 300

Problem 1. Show that $P(\mathbb{N}) \times P(\mathbb{N}) \approx P(\mathbb{N})$.

[Hint: Use the interleaving function exercise from the previous HW.]

Solution. $P(\mathbb{N}) \times P(\mathbb{N}) \sim^{(*)+(***)} {}^{\mathbb{N}}\{0,1\} \times {}^{\mathbb{N}}\{0,1\} \sim^{(**)} {}^{\mathbb{N}}\{0,1\} \sim^{(***)} P(\mathbb{N})$.

$(*)$– we saw in class that $A \sim A'$ and $B \sim B'$ then $A \times B \sim A' \times B'$.

$(**)$– the previous homework.

$(***)$– we saw in class that ${}^{\mathbb{N}}\{0,1\} \sim P(\mathbb{N})$.

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Problem 2. Prove that $P(\mathbb{Z}\times\mathbb{Z})\times P(\mathbb{Z})\sim P(\mathbb{N})$.

Solution. $P(\mathbb{Z}\times\mathbb{Z})\times P(\mathbb{Z})\sim^{(*)} P(\mathbb{N})\times P(\mathbb{Z})\sim^{(**)} P(\mathbb{N})\times P(\mathbb{N})\sim^{(***)} P(\mathbb{N})$.

$(*)-$ $\mathbb{Z}\times\mathbb{Z}\sim\mathbb{N}\times\mathbb{N}\sim\mathbb{N}$ and if $A\sim A'$ then $P(A)\sim P(A')$.

$(**)-$ $\mathbb{N}\sim\mathbb{Z}$ and if $A\sim A'$ then $P(A)\sim P(A')$.

$(***)-$ the previous exersice.

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Problem 3. Prove that if $A \sim A'$ and $B \sim B'$ are sets such that $A \cap B = A' \cap B' = \emptyset$ then $A \cup B \sim A' \cup B'$.

Solution. Let $f : A \longrightarrow A'$ be a bijection and $g : B \longrightarrow B'$ be a bijection. Define $h : A \cup B \longrightarrow A' \cup B'$ by

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

Note that h is well defined since $A \cap B = \emptyset$. To see that h is one-to-one let $x, y \in A \cup B$ be such that $h(x) = h(y)$. Let us split into cases:

(1) if $h(x) \in A'$, then since $A' \cap B' = \emptyset$, we have that $x, y \in A$ and therefore $f(x) = h(x) = h(y) = f(y)$, and since f is one-to-one, $x = y$.

(2) if $h(x) \in B'$, this is similar using he fact that g is one-to-one.

To see that h is onto, let $c \in A' \cup B'$. If $c \in A'$, since f is onto, there is $a \in A$ such that $h(a) = f(a) = c$. Similarly, if $c \in B'$ there is $a \in B$ such that $h(a) = g(a) = c$. in any case there is $x \in A \cup B$ such that $h(x) = c$ and therefore h is onto.

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Problem 4. Show that $\mathbb{R} \times \mathbb{R} \sim \mathbb{R}$

[Hint: use HW9 Problem 4]

Solution. By a theorem from class, $\mathbb{R} \sim {}^{\mathbb{N}}\{0,1\}$. Therefore

$$\mathbb{R} \times \mathbb{R} \sim^{(*)} {}^{\mathbb{N}}\{0,1\} \times {}^{\mathbb{N}}\{0,1\} \sim^{(**)} {}^{\mathbb{N}}\{0,1\} \sim \mathbb{R}$$

$(*)$ By the claim that if $A \sim A'$ and $B \sim B'$ then $A \times B \sim A' \times B'$.

$(**)$ By the Hint.

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Problem 5. Use CSB to show that every $\mathbb{N} \times (0,1) \sim \mathbb{R}$.

Solution. (Proof idea) On the one hand we have that $\mathbb{R} \sim (0,1) \preceq \mathbb{N} \times (0,1) \preceq \mathbb{R}$ The last inequality follows from the 1-1 function $f : \mathbb{N} \times (0,1) \to \mathbb{R}$ defined by $f(n,i) = n + i$.