The Ultrapower Axiom and the Galvin property

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Galvin's Theorem

In a paper by Baumgartner, Hajnal and Maté [2], the following theorem due to F. Galvin was published:

Theorem 1 (Galvin's Theorem)

Suppose that $\kappa^{<\kappa} = \kappa$. Then for every normal filter U over κ, and for any collection $\langle A_\alpha \mid \alpha < \kappa^+ \rangle \in [U]^{\kappa^+}$ consisting of κ^+-many sets, there is a subcollection $\langle A_i \mid i \in I \rangle$, of size κ (i.e. $I \in [\kappa^+]^\kappa$) such that $\bigcap_{i \in I} A_i \in U$.

In particular, if *GCH* holds and κ is a regular cardinal then from κ^+-many clubs, one can always extract κ-many for which the intersection is a club.
Let us put this combinatorical/saturation property into a definition:

Definition 2 (Galvin's Property)

Let $\mathcal{F}$ be a filter over κ and $\mu \leq \lambda$. Denote by $Gal(\mathcal{F}, \mu, \lambda)$ the following statement:

$$\forall \langle A_i \mid i < \lambda \rangle \in [\mathcal{F}]^\lambda . \exists I \in [\lambda]^\mu . \bigcap_{i \in I} A_i \in \mathcal{F}$$

Example 3

1. Galvin's Theorem $\equiv$ If $\kappa^{<\kappa} = \kappa$ the $Gal(U, \kappa, \kappa^+)$ holds for every normal U over κ.
2. If $\mu' \leq \mu \leq \lambda \leq \lambda'$ then $Gal(\mathcal{F}, \mu, \lambda) \Rightarrow Gal(\mathcal{F}, \mu', \lambda')$.
3. If (e.g.) $\mathcal{F}$ contains all the final segments and $\mu = cf(\kappa)$ then $\neg Gal(\mathcal{F}, \mu, \mu)$.

Our plan for this talk is as follows:

- $\Rightarrow$ Present a recent application of the Galvin property in the realm of the Tukey order (joint result with **N. Dobrinen**).
- $\Rightarrow$ Investigate the Galvin property under the ultrapower axiom (joint with **G. Goldberg**)

These results continue the previous investigation of the Galvin property on κ-complete ultrafilters over measurable cardinals [5, 7, 4, 9, 10] due to **Shimon Garti**, **Moti Gitik**, **Alejandro Poveda**, **Saharon Shelah** and B. .

The Tukey order

Definition 4

Let $(P, \leq_P), (Q, \leq_Q)$ be p.o's. Denote $P \leq_T Q$ if there is a *cofinal* function $f : Q \to P$, meaning that for every cofinal set $B \subseteq Q$ $(\forall q \in Q \ \exists b \in B \ q \leq_Q b)$ $f''B$ is cofinal in P. Also, $P =_T Q$ iff $P \leq_T Q \wedge Q \leq_T P$.

Proposition 1 (Schmidt)

$\exists f : Q \to P$ cofinal iff $\exists g : P \to Q$ unbounded, namely, for every unbounded set $B \subseteq P$, $g''B$ is unbounded

When U is a filter we are only considering $(U, \supseteq)$.

Example 5

- ⇒ If $U \leq_{RK} W$ then $U \leq_T W$.
- ⇒ $P \times Q = l.u.b_{\mathrm{T}}(P, Q)$.
- ⇒ $U \leq_T W$ iff $\exists f : W \to U$ *monotone*[a] and cofinal iff $\exists f : W \to U$ monotone and $Im(f)$ is cofinal in U.

[a] i.e. $A \subseteq B \Rightarrow f(A) \subseteq f(B)$.

The maximal Tukey class and the Galvin property

Definition 6

We say that a partial order P is (μ, λ)-*Tukey-top* if the Tukey class of P is maximal among μ-directed-closed posets of cardinality at most λ.

Note that μ and λ can be inferred from P and so omitted from the definition.

Theorem 7 (Dobrinen-B.)

Let U be a μ-complete ultrafilter over κ. Then U is Tukey-top (wrt. $(\mu, 2^\kappa)$) if and only if $\neg Gal(U, \mu, 2^\kappa)$.

Lemma 8

Suppose that $|P| \leq \lambda$ and P is μ-directed, then $P \leq_T ([\lambda]^{<\mu}, \subseteq)$.

Lemma 9

If U is μ-complete then $([\lambda]^{<\mu}, \subseteq) \leq_T U$ iff $\neg Gal(U, \mu, \lambda)$.

How far can we push Galvin's Theorem?

$$[\kappa^{<\kappa} = \kappa \wedge U \text{ is normal}] \Rightarrow \; Gal(U, \kappa, \kappa^+)$$

Trying to remove the assumption $\kappa = \kappa^{<\kappa}$ was proven impossible for successors of regulars by Abraham and Shelah [1] and for successors of singulars by Garti, Poveda, and B. [6]. (The question regarding (weakly) inac. cardinals remains open.)

Theorem 10 (Gitik-B.)

Suppose that $\kappa^{<\kappa} = \kappa$. Then

1. *([8] 2021) Every ultrafilter U which is Rudin-Keisler equivalent to a finite product of κ-complete p-point ultrafilters satisfies $Gal(U, \kappa, \kappa^+)$.*
2. *([3] 2023) The same for a ultra filter U which is Rudin-Keisler equivalent to a filter of the form[a]:*

$$\sum_{U}(\sum_{U_{\alpha_1}} \cdots \sum_{U_{\alpha_1,..,\alpha_{n-1}}}(U_{\alpha_1,\dots,\alpha_n})\dots)$$

*where U and each $U_{\alpha_1,\dots,\alpha_k}$ are p-point ultrafilters. Such an ultrafilter called an n-*fold sum of *p-*points*.*

[a] Suppose that W is an ultrafilter over κ and W_α is an ultrafilter over $\delta_\alpha \leq \kappa$. Then $\sum_W W_\alpha = \{X \subseteq [\kappa]^2 \mid \{\alpha < \kappa \mid \{\beta \mid \langle \alpha, \beta \rangle \in X\} \in W_\alpha\} \in W\}$.

Our results imply in particular that the theorem from the previous slide is optimal.

Theorem 11 (B. [3])

Suppose that there is no inner model with a superstrong cardinal, then if $L[E]$ is an iterable Mitchell-steel model, every κ-complete ultrafilter is an n-fold sum of p-points and in particular has the Galvin property.

Theorem 12 (Goldberg-B. [11])

Assume UA plus every irreducible is Dodd sound, then for every κ-complete ultrafilter U over κ, U has the Galvin property iff U is an n-fold sum of p-points.

The assumptions of the theorem holds if $L[E]$ is Mitchell-Steel iterable as proven by Goldberg and Schlutzenberg.

Theorem 13 (Gitik [?])

Starting from a measurable cardinal, it is consistent that there is an ultrafilter U with the Galvin property which is not an n-fold sum of p-points.

The proof

Diamond-like principles

Definition 14

Let U be a σ-complete ultrafilter over κ. We say that $\diamondsuit^-_{\text{thin}}(U)$ holds iff there is $A \in M_U$ and $\lambda < j_U(\kappa)$ such that:

1. $\{j_U(S) \cap \lambda \mid S \in P(\kappa)\} \subseteq A$.
2. For every $f : \kappa \to \kappa$, $j_U(f)(|A|^{M_U}) < \lambda$.

If $A = [\alpha \mapsto A_\alpha]_U$ and $\lambda = [f_\lambda]_U$, then the first bullet says that $\langle A_\alpha \mid \alpha < \kappa \rangle$ is a guessing sequence modulo U, for subsets of $S \subseteq \kappa$ and the guessing appears and $S \cap f_\lambda(\alpha)$. The second condition says that the cardinality of A_α grows in a relatively controlled way. More accurately, in Kanamori's language of "skies and constellations", the cardinality of A should be in a lower "sky" then λ.

Definition 15

An ultrafilter U is λ-sound if the map $j^\alpha : P(\kappa) \to M_U$ defined by $j^\alpha(S) = j_U(S) \cap \alpha$ is in M_U. In particular $\{j_U(S) \cap \alpha \mid S \in P(\kappa)\} \in U$. U is called Dodd-sound if it is $[id]_U$-sound.

Clearly, normal ultrafilters are Dodd sound.

Proposition 2

If U is λ-sound for λ such that for all $f : \kappa \to \kappa$, $j_U(f)(\kappa) < \lambda$, then $\diamondsuit^-_{thin}(U)$ holds.

Definition 16

U is called a *p*-point ultrafilter over κ if and only if every function $f : \kappa \to \kappa$ which is unbounded mod U, is almost one-to-one mod U i.e. there is a set $X \in U$ such that for every $\alpha < \kappa$, $\sup(f^{-1}[\alpha] \cap X) < \kappa$.

For κ-complete ultrafilters over κ, this is equivalent to the assertion that there is a function f such that $j_U(f)(\kappa) \geq [id]_U$.

Corollary 17

U is non p-point which is Dodd-sound then $\diamondsuit^-_{thin}(U)$ holds.

The relevance of this principle to the Galvin property is the following:

Theorem 18

$\diamondsuit^{-}_{thin}(U)$ implies $\neg Gal(U, \kappa, 2^{\kappa})$.

Proposition 3

Suppose $U \leq_{RK} W$. $\diamondsuit^{-}_{thin}(U) \Rightarrow \diamondsuit^{-}_{thin}(W)$.

Proposition 4

Suppose that U is an ultrafilter on $\lambda \leq \kappa$ and $\langle W_\xi \mid \xi < \lambda \rangle$ is a sequence of ultrafilters over κ such that for every ξ, $\diamondsuit^{-}_{thin}(W_\xi)$, then $\diamondsuit^{-}_{thin}(\sum_U W_\xi)$.

The Ultrapower Axiom

The Ultrapower Axiom and its associated theory was developed by G. Goldberg and can be viewed as an attempt to formalize the notion of a canonical inner model with a single simple axiom. It follows from weak comparison and therefore holds in every known canonical inner model.

Definition 19

The Ultrapower Axiom (UA) is the assertion that for every two σ-complete ultrafilters U, W, there are σ-complete ultrafilters $W^* \in M_U$ and $U^* \in M_W$ such that $(M_{U^*})^{M_W} = (M_{W^*})^{M_U}$ and $j_{U^*} \circ j_W = j_{W^*} \circ j_U$.

It seems that the ultrapower axiom determines completely the structure of the σ-complete ultrafilters. For example, it implies that the Mitchell order is linear, that the Ketonen order is linear and, more relevant for our purposes, every ultrapower embedding factorizes canonically.

Definition 20

For two ultrafilters U, W over X, Y (resp.) we say that $U \leq_{RF} W$ if there is a sequence of ultrafilters, $\langle W_\xi \mid \xi < \kappa \rangle$ such that $W =_{RK} \sum_U W_\xi$.

This is equivalent to the requirement that there is an internal ultrapower embedding (by $[\xi \mapsto W_\xi]_U$) such that.

Definition 21

An ultrafilter U is *irreducible* if it is *RF*-minimal among non-principal ultrafilters. Equivalently, there is no ultrapower embedding which factors j_U using an internal ultrapower.

normal ultrafiters and Dodd sound ultrafilters are irreducible.

Theorem 22 (Goldberg (UA))

For every σ-complete ultrafilter U, every ascending sequence of ultrafilters $D_0 <_{RF} D_1 <_{RF} D_2 ... \leq_{RF} U$ is finite.

Theorem 23 (Goldbeg-B. (UA))

Assume that every irreducible is Dodd-sound. If W is a κ-complete ultrafilter over κ, then the following are equivalent:

1. *W has the Galvin property.*
2. $\neg\diamondsuit^-_{thin}(W)$.
3. *W is an n-fold sum of κ-complete p-points over κ*

Proof (Almost...)

- $\Rightarrow$ From previous results, (3) $\Rightarrow$ (1) $\Rightarrow$ (2). We shall prove that (2) $\Rightarrow$ (3) and let W be an ultrafilter which is not an n-fold sum of κ-complete p-points.
- $\Rightarrow$ There is $U \leq_{RF} W$ which is RF-maximal and is an n-fold sum of κ-complete p-points. (Note that there is such a p-point RF-below W. Indeed, any irreducible (and thus Dodd sound) can be assumed to be p-point since otherwise it is a non p-point Dodd sound).
- $\Rightarrow$ Let $\langle W_\xi \mid \xi < \kappa\rangle$ be a sequence of ultrafilters over κ such that $W = \sum_U W_\xi$. Let $D_\xi \leq_{RF} W_\xi$ be irreducible ultrafilter over δ_ξ which is ρ_ξ-complete ($\rho_\xi \leq \delta_\xi \leq \kappa$). It suffices to prove that $\diamondsuit^-_{\text{thin}}(\sum_U D_\xi)$ holds.

By our choice, $j_U : V \to M_U$ can be factored as an iterated ultrapower

$$V = M_0 \xrightarrow{j_{0,1}} M_1 \xrightarrow{j_{1,2}} \dots \xrightarrow{j_{n-1,n}} M_n = M_U$$

where in M_k, $j_{k,k+1}$ is the ultrapower by a κ_k-complete p-point U_k over $\kappa_k = j_{0,k}(\kappa)$. Let $Z_k = \sum_{U_0} \dots \sum_{U_{k-2}} U_{k-1}$ be the ultrafilter associate with $j_{0,k}$. Find m such that $\kappa_{m-1} < \delta^* := [\xi \mapsto \delta_\xi]_U \leq \kappa_m$ and $D^* = [\xi \mapsto D_\xi]_U$ is an M_U-ultrafilter over δ^*. Since $crit(j_{m,n}) = \kappa_m$ it is an M_m-ultrafilter and $(j_{D^*})^{M_m} \restriction M_n = (j_{D^*})^{M_n}$. By elementarity of $j^{M_m}_{D^*}$, $j^{M_n}_{D^*} \circ j_{m,n} = j^{M_m}_{D^*}(j_{m,n}) \circ j^{M_m}_{D^*}$

$$j^{M_n}_{D^*} \circ j_U = j^{M_m}_{D^*}(j_{m,n}) \circ j^{M_m}_{D^*} \circ j_{0,m}.$$

- $\Rightarrow$ If $M_m \models D^*$ is a κ_m-complete ultrafilter over κ_m (in particular if $m = 0$ this is the case), then D^* must be a non p-point (and Dodd sound) by the maximality of U (as $U <_{RF} \sum_U D_\xi \leq_{RF} W$). It follows that $M_m \models \diamondsuit^-_{\text{thin}}(D^*)$ and therefore $\diamondsuit^-_{\text{thin}}(\sum_{Z_m} D^*)$ holds. But this ultrafilter is RK below W and therefore $\diamondsuit^-_{\text{thin}}(W)$ holds. So we may assume that D^* is not κ_m-complete. **We omit the case where $\delta^* = \kappa_m$.** and assume that $\delta^* < \kappa$.
- $\Rightarrow$ Assume $crit(j^{M_m}_{D^*}) > \kappa_{m-1}$. Note that the two step iteration $j^{M_m}_{D^*} \circ j_{U_{m-1}}$ is given by a κ_{m-1}-complete p-point on κ_{m-1} in M_{m-1}, which contradicts the maximality of U.
- $\Rightarrow$ Assume $crit(j^{M_m}_{D^*}) \leq \kappa_{m-1} < \delta^*$. It is a consequence of UA [12, Theorem 8.2.22] that the irreducibility of D^* implies M_{D^*} is closed under κ_{m-1}-sequences which in turn implies that $P(\kappa_{m-1}) \subseteq (M_{D^*})^{M_m}$. By the Kunen inconsistency Theorem, $j^{M_m}_{D^*}(\kappa_{m-1}) > \kappa_{m-1}$. Let $\lambda = j^{M_m}_{D^*}(\kappa_{m-1}$. We claim that $\sum_{U_{m-1}} D^*$ is λ-sound and that for every function $f : \kappa_{m-1} \to \kappa_{m-1}$, $j_{\sum_{U_{m-1}} D^*}(f)(\kappa_{m-1}) < \lambda$ which then implies $M_{m-1} \models \diamondsuit^-_{\text{thin}}(\sum_{U_{m-1}} D^*)$. Thus $\diamondsuit^-_{\text{thin}}(\sum_{Z_{m-1}}(\sum_{U_{m-1}} D^*))$ and this ultrafilter is Rudin-Keisler below W.

⇒ For any function $f : \kappa_{m-1} \to \kappa_{m-1}$, since $j_{D^*}(\kappa_{m-1}) > \kappa_{m-1}$, $j_{D^*}(j_{U_{m-1}}(f))(\kappa_{m-1}) = j_{D^*}(j_{U_{m-1}}(f) \restriction \kappa_{m-1})(\kappa_{m-1}) = j_{D^*}(f)(\kappa_{m-1})$, and $j_{D^*}(f) : j_{D^*}(\kappa_{m-1}) \to j_{D^*}(\kappa_{m-1})$. We have that $j_{D^*}(f)(\kappa_{m-1}) < j_{D^*}(\kappa_{m-1})$.

⇒ To see that $\sum_{U_{m-1}} D^*$ is λ-sound, derive the (κ_{m-1}, λ)-extender E from $j_{D^*}^{M_m}$ inside M_m. Note that E is also the (κ_{m-1}, λ)-extender derived from $j_{D^*}^{M_m} \circ j_{m-1,m}$ since for $\alpha < j_{D^*}(\kappa_{m-1})$ we have that:
$\alpha \in j_{D^*}(j_{m-1,m}(X)) \cap j_{D^*}(\kappa_{m-1})$ iff $\alpha \in j_{D^*}(j_{m-1,m}(X) \cap \kappa_{m-1})$ iff $\alpha \in j_{D^*}(X)$.
Now since D^* is a (uniform) ultrafilter over δ^*, $j_{D^*}(\kappa_{m-1})$ and since it is Dodd sound, we have that $E \in (M_{D^*})^{M_m}$. Also, $(M_{D^*})^{M_m}$ is closed under κ_{m-1}-sequences from M_m, hence $\{j_E(X) \mid X \subseteq \kappa_{m-1}\} \in (M_{D^*})^{M_m}$ where $j_E : M_{m-1} \to M_E$. Let $k_E : M_E \to (M_{D^*})^{M_m}$ be the factor map, note that $crit(k_E) = j_{D^*}(\kappa_{m-1})$. Note that $j_{\sum_{U_{m-1}} D^*}(X) \cap j_{D^*}(\kappa_{m-1}) = j_E(X)$, hence

$$\{j_{\sum_{U_{m-1}} D^*}(X) \cap j_{D^*}(\kappa_{m-1}) \mid X \subseteq \kappa_{m-1}\} \in (M_{D^*})^{M_m}$$

as desired.

Further Results

Theorem 24 (UA)

Assume that every irreducible ultrafilter is Dodd sound. For every σ-complete ultrafilter W over κ the following are equivalent:

1. *W has the Galvin property.*
2. $\neg\diamondsuit^{-}_{thin}(W)$.
3. *W is the D-sum of n-fold sums of κ-complete p-points over κ and D is a σ-complete ultrafilter on $\lambda < \kappa$.*

Theorem 25 (UA)

Assume that every irreducible ultrafilter is Dodd sound. Suppose κ is an uncountable cardinal that carries a κ-complete non-Galvin ultrafilter. Then the Ketonen least non-Galvin κ-complete ultrafilter on κ extends the closed unbounded filter.

Definition 26

κ is called *non-Galvin cardinal* if there are elementary embeddings $j\colon V \to M$, $i\colon V \to N$, $k\colon N \to M$ such that:

1. $k \circ i = j$.
2. $crit(j) = \kappa$, $crit(k) = i(\kappa)$.
3. ${}^{\kappa}N \subseteq N$ and ${}^{\kappa}M \subseteq M$
4. there is $A \in M$ such that $i''\kappa^+ \subseteq A$ and $M \models |A| < i(\kappa)$.

Theorem 27

Suppose that κ is a non-Galvin cardinal. Then there exists a κ-complete ultrafilter U over κ such that $\neg Gal(U, \kappa, \kappa^+)$.

Theorem 28 (UA)

Assume that every irreducible ultrafilter is Dodd sound. If there is a κ-complete non-Galvin ultrafilter on an uncountable cardinal κ, then there is a non-Galvin cardinal.

Thank you for your attention!

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