ON THE TUKEY TYPES OF FUBINI PRODUCTS

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Abstract. We extend the class of ultrafilters U over countable sets for which $U \cdot U \equiv_T U$, extending several results from [13]. In particular, we prove that for each countable ordinal $\alpha \geq 2$, the generic ultrafilter G_α forced by $P(\omega^\alpha)/\text{fin}^{\otimes\alpha}$ satisfy $G_\alpha \cdot G_\alpha \equiv_T G_\alpha$. This answers a question posed in [13, Question 43]. Additionally, we establish that Milliken-Taylor ultrafilters possess the property that $U \cdot U \equiv_T U$.

0. Introduction

The Tukey order of partially ordered sets finds its origins in the notion of Moore-Smith convergence [31], which generalizes the usual meaning of convergence of sequence to *net*, allowing to enlarge the class of topological spaces for which continuity is equivalent to continuity in the sequential sense. Formally, given two posets, $(P, \leq_P)$ and $(Q, \leq_Q)$ we say that $(P, \leq_P) \leq_T (Q, \leq_Q)$ if there is map $f : Q \to P$, which is cofinal, namely, $f''B$ is cofinal in P whenever $B \subseteq Q$ is cofinal. Schmidt [28] observed that this is equivalent to having a map $f : P \to Q$, which is unbounded, namely, $f''\mathcal{A}$ is unbounded in Q whenever $\mathcal{A} \subseteq P$ is unbounded in P. We say that P and Q are *Tukey equivalent*, and write $P \equiv_T Q$, if $P \leq_T Q$ and $Q \leq_T P$; the equivalence class $[P]_T$ is called the Tukey type or cofinal type of P. It turns out that the Tukey order restricted to posets $(U, \supseteq)$, where U is an ultrafilter, has a close relation to *ultranets* and has been studied extensively on ω by Blass, Dobrinen, Kuzeljevic, Milovich, Raghavan, Shelah, Todorcevic, Verner, and others (see for instance [5, 8, 12, 13, 21, 23, 25, 27]). Recently, the authors extended this investigation to the realm of large cardinals where they considered the Tukey order on σ-complete ultrafilters over a measurable cardinal κ in [2]. On ultrafilters, the Tukey order is determined by functions which are (weakly) monotone[1] and have cofinal images. For this reason, the Tukey order is the order one expects to use when comparing the cofinality of ultrafilters. We refer the reader to [7] and [10] for surveys of the subject.

In this paper, we investigate the connection between the Tukey type of an ultrafilter U and the Tukey type of its Fubini product with itself, $U \cdot U$. It is easy to see that $U \leq_T U \cdot U$ for every ultrafilter U. The question is, for which ultrafilters

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[1]That is, $a \leq b \Rightarrow f(a) \leq f(b)$.

does $U \equiv_T U \cdot U$ hold? Already Dobrinen and Todorcevic [13] and Milovich [24], provide a significant understanding of the relation between these two Tukey types: Dobrinen and Todorcevic proved that whenever U is a rapid p-point ultrafilter over ω, then $U \cdot U \equiv_T U$; Milovich proved that $U \cdot U \cdot U \equiv_T U \cdot U$ for every nonprincipal ultrafilter U. In contrast, trivial examples of ultrafilters U with the property that $U \cdot U \equiv_T U$ are the so-called *Tukey-top* ultrafilters, those ultrafilters which are maximal in the Tukey order among all ultrafilters on ω. Such an ultrafilter was constructed (in ZFC) by Isbell [18] and Juhász [19]; we denote this ultrafilter by $\mathcal{U}_{\text{top}}$. Henceforth, we shall only focus on nonprincipal ultrafilters U such that $U <_T \mathcal{U}_{\text{top}}$.

Dobrinen and Todorcevic also proved that for p-points, $U \cdot U \equiv_T U$ is equivalent to U being Tukey above $(\omega^\omega, \leq)$, where $\leq$ refers to the everywhere domination order of functions. Furthermore, they provided an example of a p-point U which is not Tukey above ω^ω, and in particular satisfying $U <_T U \cdot U$. They also asked whether, besides the Tukey-top ultrafilters, the class of ultrafilters which are Tukey equivalent to their Fubini product is a subclass of the class of basically generated ultrafilters.

Question 0.1. [13, Q.43] Does $U \cdot U \equiv_T U < \mathcal{U}_{\text{top}}$ imply that U is basically generated?

Note that on measurable cardinals, the situation is quite different, as every κ-complete ultrafilter U over κ satisfies that $U \cdot U \equiv_T U$ [2, Thm 5.6].

In this paper, we provide a negative answer to this question by analyzing the Tukey type of the Fubini powers of generic ultrafilters obtained by σ-closed forcings of the form $P(X)/I$. We define the *I-pseudo intersection property* (Definition 1.11) as a way of abstracting the notion of p-point, and use it to extend work in [13] and [24] to provide an abstract condition which guarantees that $U \cdot U \equiv_T U$ (Corollary 1.17). We then provide an equivalent formulation for U being Tukey equivalent to itself (Theorem 1.18), generalizing [13, Thm. 35].

In Section 2, we apply Theorem 1.18 to the forcing $P(\omega \times \omega)/I$, where $I = \text{fin} \otimes \text{fin} := \text{fin}^{\otimes 2}$ over $\omega \times \omega$. This forcing was first investigated by Szymánski and Zhou [29] and later by many others. We show that any generic ultrafilter G_2 for $P(\omega \times \omega)/\text{fin}^{\otimes 2}$ has the property that $G_2 \cdot G_2 \equiv_T G_2$. Blass, Dobrinen and Raghavan [5] showed that G_2 is not basically generated (so in particular is not a p-point) but also is not Tukey-top. Dobrinen proved in [8] that G_2 is in fact the Tukey immediate successor of its projected Ramsey ultrafilter, so at the same level as a weakly Ramsey ultrafilter (see [14]) in the Tukey hierarchy.

In Section 3, we investigate ultrafilters G_α obtained by forcing $P(\omega^\alpha)/\text{fin}^{\otimes\alpha}$, for any $2 \leq \alpha < \omega_1$. We analyze the Tukey type of $\text{fin}^{\otimes\alpha}$ and prove that, for all $\alpha < \omega_1$, $G_\alpha \equiv_T G_\alpha \cdot G_\alpha$ (Theorem 3.9). Such ultrafilters are not p-points and also not basically generated, and each G_α Rudin-Keisler projects onto generic ultrafilters G_β for $\beta < \alpha$. By results of Dobrinen [8, 9, 6], such ultrafilters are non-Tukey top, and moreover, are quite low in the Tukey hierarchy. For instance, for $k < \omega$, the sequence $\langle G_n \mid n \leq k \rangle$ form an exact Tukey chain in the sense that if $V \leq_T G_k$, then there is $n \leq k$, $G_n \equiv_T V$. Related results holds for $\omega \leq \alpha < \omega_1$.

Finally, in Section 4, we prove that if U is a Milliken-Taylor ultrafilter then also U satisfies our equivalent condition and therefore $U \cdot U \equiv_T U$ (Theorem 4.8). Milliken-Taylor ultrafilters as well as those forced by $P(\omega^\alpha)/\text{fin}^{\otimes\alpha}$ are not basically generated and in particular p-points, providing examples which answer Question 0.1 in the negative.

One question which our results are relevant for but remains open, is whether $U\cdot V\equiv_T V\cdot U$ for any ultrafilter U,V over ω. Corollaries 1.9 and 1.10 provide some progress on this question. This again contrasts with ultrafilters on a measurable cardinal κ: work of the authors in [2] showed that if U,V are κ-complete ultrafilters over κ, then $U\cdot V\equiv_T V\cdot U$. The proof essentially uses the well-foundedness of ultrapowers by κ-complete ultrafilters and therefore does not apply for ultrafilters over ω. This is discussed in Section 1.

0.1. **Notation.** $[X]^{<\lambda}$ denotes the set of all subsets of X of cardinality less than λ. Let $\text{fin}=[\omega]^{<\omega}$, and $\text{FIN}=\text{fin}\setminus\{\emptyset\}$. For a collection of sets $(P_i)_{i\in I}$ we let $\prod_{i\in I}P_i=\{f: I\to\bigcup_{i\in I}P_i\mid \forall i,\ f(i)\in P_i\}$. Given a set $X\subseteq\omega$, such that $|X|=\alpha\leq\omega$, we denote by $\langle X(\beta)\mid\beta<\alpha\rangle$ be the increasing enumeration of X. Given a function $f:A\to B$, for $X\subseteq A$ we let $f''X=\{f(x)\mid x\in X\}$ and for $Y\subseteq B$ we let $f^{-1}Y=\{x\in X\mid f(x)\in Y\}$. Given sets $\{A_i\mid i\in I\}$ we denote by $\biguplus_{i\in I}A_i$ the union of the A_i's when the sets A_i are pairwise disjoint.

1. The Tukey type of a Fubini product

Given $\mathcal{A}\subseteq P(X)$, we set $\mathcal{A}^*=\{X\setminus A\mid A\in\mathcal{A}\}$. For a filter F over X, we denote the *dual ideal* by F^*, and given an ideal I we denote the *dual filter* by I^*. The following fact is easy to verify:

Fact 1.1. *For any filter F, $(F,\supseteq)\equiv_T(F^*,\subseteq)$.*

An ultrafilter over X is a filter U such that for every $A\in P(X)$, either $A\in U$ or $X\setminus A\in U$. So for ultrafilters we have that $U^*=P(X)\setminus U$. As the title of this section indicates, we are interested in the Fubini product of ultrafilters:

Definition 1.2. Suppose that U is a filter over X and for each $x\in X$, U_x is a filter over Y_x. We denote by $\sum_U U_x$ the filter over $\bigcup_{x\in X}\{x\}\times Y_x$, defined by

$$A\in\sum_U U_x \text{ if and only if } \{x\in X\mid (A)_x\in U_x\}\in U$$

where $(A)_x=\{y\in Y_x\mid\langle x,y\rangle\in A\}$. If for every x, $U_x=V$ for some fixed V over a set Y, then $U\cdot V$ is defined as $\sum_U V$, which is a filter over $X\times Y$. U^2 denotes the filter $U\cdot U$ over $X\times X$.

It is well known that if U and each U_x are ultrafilters, then also $\sum_U U_x$ is an ultrafilter (see for example see [3]).

Fact 1.3. *If U,V are filters over countable sets X,Y respectively, then $U\equiv_{RK}U'$, $V\equiv_{RK}V'$ for some filters U',V' over ω and $U\cdot V\equiv_{RK}U'\cdot V'$.*

Given posets $(P_i,\leq_i)_{i\in I}$ we let $\prod_{i\in I}(P_i,\leq_i)$ be the ordered set $\prod_{i\in I}P_i$ with the pointwise order derived from the orders $\leq_i$. We call this order on $\prod_{i\in I}P_i$, the *everywhere domination order*. We will omit the order when it is the natural order. In particular, any ordinal α is ordered naturally by $\in$, $\omega^\omega=\prod_{n<\omega}\omega$ is ordered by everywhere domination, and $\prod_{n<\omega}\omega^\omega$ is the everywhere domination order where each ω^ω is again ordered by everywhere domination.

Fact 1.4. *If A, B are any sets with the same cardinality and $(P, \preceq)$ is an ordered set, then $\prod_{a\in A}(P,\preceq)$ and $\prod_{b\in B}(P,\preceq)$ equipped with the everywhere domination orders are order isomorphic.*

The next theorem of Dobrinen and Todorcevic provides an upper bound for the Fubini product of ultrafilters via the Cartesian product.

Theorem 1.5 (Dobrinen-Todorcevic, Thm. 32, [13])**.** $\sum_U U_x \leq_T U \times \prod_{x\in X} U_x$. *In particular,* $U \cdot V \leq_T U \times \prod_{x\in X} V$ *and* $U \cdot U \leq_T \prod_{x\in X} U$.

Towards answering a question from [13], Milovich improved the previous theorem over ω. Let us give a slight variation of his proof, for we will need it later:

Proposition 1.6 (Milovich, Lemma 5.1, [24])**.** *For filters U, V over countable sets X, Y (resp.),* $U \cdot V \equiv_T U \times \prod_{x\in X} V$. *In particular* $U \cdot U \equiv_T \prod_{x\in X} U$.

Remark 1.7. We cannot prove in general that $\sum_U U_n \equiv_T U \times \prod_{n<\omega} U_n$. For example, if U is such that $U\cdot U \equiv_T U$ (e.g., U is Ramsey) and U_0 is Tukey-top while $U_n = U$ for every $n > 0$, then $\sum_U U_n = U\cdot U \equiv_T U <_T U_0$ but $\prod_{n<\omega} U_n \geq_T U_0$, so we have $\sum_U U_n \not\equiv_T U \times \prod_{n<\omega} U_n$. Similar examples can be constructed even when all the U_n's are distinct, for example, requiring that for every $n > 0$, $U_n \leq_T U$ for some U such that $U \cdot U \equiv_T U$. Such examples are constructed in this paper in Section 3: Take U to be a generic ultrafilter for $P(\omega^\omega)/\mathrm{fin}^{\otimes\omega}$, and U_n the Rudin-Keisler projection of U to a generic on $P(\omega^n)/\mathrm{fin}^{\otimes n}$.

Proof. The proof of Theorem 1.5 does not use the fact that the partial orders are ultrafilters, and we have that $U \cdot V \leq_T U \times \prod_{x\in X} V$. (Indeed, the map $F : U \times \prod_{x\in X} V \to U \cdot V$ defined by $F(A, \langle Y_x \mid x \in X\rangle) = \bigcup_{x\in A}\{x\} \times Y_x$ is monotone and cofinal). For the other direction, by Facts 1.3 and 1.4, we may assume that $X = Y = \omega$. Let us define a cofinal map from a cofinal subset of $U\cdot V$ to $U\times\prod_{n<\omega} V$. Consider the collection $\mathcal{X} \subseteq U \cdot V$ of all $A \subseteq \omega \times \omega$ such that:

(1) for all $n < \omega$, either $(A)_n = \emptyset$ or $(A)_n \in V$.
(2) $\pi'' A \in U$ and for all $n_1, n_2 \in \pi_1'' A$, if $n_1 < n_2$ then $(A)_{n_2} \subseteq (A)_{n_1}$.

It is not hard to prove that $\mathcal{X}$ is a filter base for $U \cdot V$. Let $F : \mathcal{X} \to U \times \prod_{n<\omega} V$ be defined by $F(A) = \langle \pi'' A, \langle (A)_{(\pi_1'' A)(n)} \mid n < \omega\rangle\rangle$ where $\langle(\pi_1'' A)(n) \mid n < \omega\rangle$ is the increasing enumeration of $\pi_1'' A$. Let us prove that F is monotone and cofinal. Suppose that $A, B \in \mathcal{X}$ are such that $A \subseteq B$. Then

a. $\pi'' A \subseteq \pi'' B$;
b. for every $n < \omega$, $(\pi'' A)(n) \geq (\pi'' B)(n)$;
c. for every $m < \omega$, $(A)_m \subseteq (B)_m$.

By requirement (2) of sets in $\mathcal{X}$, for every $n < \omega$,

$$(A)_{(\pi'' A)(n)} \subseteq (B)_{(\pi'' A)(n)} \subseteq (B)_{(\pi'' B)(n)}.$$

It follows that $F(A) \geq F(B)$. To see that F is cofinal, let $\langle B, \langle B_n \mid n < \omega\rangle\rangle \in U \times \prod_{n<\omega} V$. Define $A = \bigcup_{n\in B}\{n\} \times (\bigcap_{m\leq n} B_m)$. Then $\pi'' A = B$ and it is straightforward that $A \in \mathcal{X}$. We claim that for every n, $F(A)_n \subseteq B_n$. Indeed, $B(n) = (\pi'' A)(n) \geq n$ and therefore $F(A) = \langle B, \langle A_n \mid n < \omega\rangle\rangle$ where $A_n = \bigcap_{m\leq B(n)} B_m \subseteq B_n$. □

Milovich used this proposition to deduce the following, answering a question in [13] and improving a result in [13] which showed that (2) below holds if F and G are both rapid p-points:

Theorem 1.8 (Milovich, Thms. 5.2, 5.4, [24]). (1) *For any filters* F, G, $F \cdot G \cdot G \equiv_T F \cdot G$. *In particular,* $F^{\otimes 2} \equiv_T F^{\otimes 3}$.

(2) *If* F, G *are p-filters, then* $F \cdot G \equiv_T G \cdot F$.

For the rest of this section, let us derive several corollaries and present new results.

Corollary 1.9. *Suppose that* $V \cdot V \equiv_T V$. *Then* $U \cdot V \equiv_T U \times V$. *Moreover, if also* $U \cdot U \equiv_T U$, *then* $U \cdot V \equiv_T V \cdot U$.

Proof. Since $V \cdot V \equiv_T V$, we have that $V \equiv_T \prod_{n<\omega} V$, hence

$$U \cdot V \equiv_T U \times \prod_{n<\omega} V \equiv_T U \times V.$$

For the second part, it is clear that $U \times V \equiv_T V \times U$ and therefore if $V \cdot V \equiv_T V$ and $U \cdot U \equiv_T U$, then $U \cdot V \equiv_T V \cdot U$. □

Corollary 1.10. *For any ultrafilters* U, V *on countable sets,* $U \cdot V \cdot V \equiv_T U \times (V \cdot V)$. *In particular* $(U \cdot U) \cdot (V \cdot V) \equiv_T (V \cdot V) \cdot (U \cdot U)$.

As a consequence of Proposition 1.6, $U \cdot U \equiv_T U$ if and only if $U \equiv_T \prod_{n<\omega} U$. Still, checking whether U is Tukey equivalent to $\prod_{n<\omega} U$ is usually a non-trivial task. We provide below a simpler condition and explain how it generalizes some results from [13].

Definition 1.11. Let U be an ultrafilter over a countable set X and $I \subseteq U^*$ an ideal on X. We say that U *has the* I*-pseudo intersection property* (abbreviated by I-p.i.p.) if for any sequence $\langle A_n \mid n < \omega \rangle \subseteq U$ there is a set $A \in U$ such that for every $n < \omega$, $A_n \subseteq^I A$, namely, $A_n \setminus A \in I$.

This definition is a generalization of being a p-point as the following example suggests.

Example 1.12. U having the fin-p.i.p. is equivalent to U being a p-point.

Claim 1.13. *For every ultrafilter* U *over* X, U^**-p.i.p. holds.*

Proof. For any sequence $\langle A_n \mid n < \omega \rangle$ we can always take $A = X \in U$, since then $A \setminus A_n = X \setminus A_n \in U^*$ by definition. □

Proposition 1.14. *If* U *has the* I*-p.i.p., then*

$$U \cdot U \leq_T U \times \prod_{n<\omega} I$$

Proof. Let U be any ultrafilter. Then by Proposition 1.6, $U \cdot U \equiv_T \prod_{n<\omega} U$. We claim that if U has the I-p.i.p., then $\prod_{n<\omega} U \leq_T U \times \prod_{n<\omega} I$. Let $\langle A_n \mid n < \omega \rangle \in \prod_{n<\omega} U$, and choose $A \in U$ such that $A \setminus A_n \in I$, which exists by I-p.i.p. Define $F(\langle A_n \mid n < \omega \rangle) = \langle A, \langle A \setminus A_n \mid n < \omega \rangle \rangle \in U \times \prod_{n<\omega} I$. We claim that F is unbounded. Indeed, suppose that $\mathcal{A} \subseteq \prod_{n<\omega} U$ and $F''\mathcal{A}$ is bounded by $\langle A^*, \langle X_n^* \mid n < \omega \rangle \rangle \in U \times \prod_{n<\omega} I$. Define $A_n^* = A^* \setminus X_n^*$. Note that $I \subseteq U^*$ and $A^* \in U$ imply that $\langle A_n^* \mid n < \omega \rangle \in \prod_{n<\omega} U$. Let us show that this is a bound for $\mathcal{A}$. Let $\langle A_n \mid n < \omega \rangle \in \mathcal{A}$. Then $F(\langle A_n \mid n < \omega \rangle \in \mathcal{A}) = \langle A, \langle A \setminus A_n \mid n < \omega \rangle \rangle \leq \langle A^*, \langle X_n^* \mid n < \omega \rangle \rangle$, namely, $A^* \subseteq A$ and $A \setminus A_n \subseteq X_n^*$. It follows that $A_n^* = A^* \setminus X_n^* \subseteq A \setminus (A \setminus A_n) = A \cap A_n \subseteq A_n$. Hence $\langle A_n \mid n < \omega \rangle \leq \langle A_n^* \mid n < \omega \rangle$, as desired. □

Corollary 1.15. *If U is a p-point then $U \cdot U \leq_T U \times \prod_{n<\omega} \text{fin}$.*

Fact 1.16. $(\text{fin}, \subseteq) \equiv_T (\omega, \leq)$.

Proof. The collection of all sets of the form $\{0, ..., n\}$ is cofinal in fin and is clearly order isomorphic to ω. □

Translating Fact 1.16 to $\prod_{n<\omega} \text{fin}$, we see that $\prod_{n<\omega} \text{fin} \equiv_T (\omega^\omega, \leq)$, where ω^ω is the set of all functions $f : \omega \to \omega$ and the order $\leq$ refers to the everywhere domination order. We conclude that if U is a p-point then $U \cdot U \leq_T U \times \omega^\omega$ (this is the important part of [13, Thm. 33]). We can now derive the following sufficient condition:

Corollary 1.17. *Let U be an ultrafilter over a countable set X, $I \subseteq U^*$ an ideal on X and*

(1) *U has the I-p.i.p., and*
(2) *$\prod_{n<\omega} I \leq_T U$.*

Then $U \cdot U \equiv_T U$.

Proof. $U \leq_T U \cdot U \leq_T U \times \prod_{n<\omega} I \leq_T U \times U \equiv_T U$. □

The above sufficient condition is in fact an equivalence:

Theorem 1.18. *For every ultrafilter U over a countable set X, the following are equivalent:*

(1) *$U \cdot U \equiv_T U$.*
(2) *$\prod_{n<\omega} U \equiv_T U$.*
(3) *There is an ideal I such that I-p.i.p. holds and $\prod_{n<\omega} I \leq_T U$.*

Proof. (1) and (2) are equivalent by Proposition 1.6. (2) $\Rightarrow$ (3) is trivial, taking $I = U^*$ and by Claim 1.13. Finally, (3) $\Rightarrow$ (1) follows from the previous corollary. □

In particular, if U is a p-point, the above proposition provides the equivalence that $U \cdot U \equiv_T U$ if and only if $U \geq_T \omega^\omega$, recovering [13, Thm. 35].

Recall that an ultrafilter U over ω is *rapid* if for every increasing function $f : \omega \to \omega$ there is $X \in U$ such that for every $n < \omega$, $otp(X \cap f(n)) \leq n$.

Fact 1.19. *U is rapid if and only if the following map is cofinal: $F : U \to \omega^\omega$ defined by $F(X) = \langle X(n) \mid n < \omega \rangle$, where $X(n)$ is the n^{th} element of X.*

As a corollary, we obtain once more a result from [13]:

Corollary 1.20. *If U is a rapid p-point then $U \equiv_T U \cdot U$.*

By taking ideals other than fin, we will find ultrafilters that are not p-points but are Tukey equivalent to their Fubini product.

2. The ideal fin $\otimes$ fin

Let I, J be ideals on X, Y (resp.). We define the Fubini product of the ideals $I \otimes J$ over $X \times Y$: For $A \subseteq X \times Y$,

$$A \in I \otimes J \text{ iff } \{x \in X \mid (A)_x \notin J\} \in I.$$

We note that this is the dual operation of the Fubini product of filters:

Fact 2.1. *For every two ideals I, J, $(I \otimes J)^* = I^* \cdot J^*$.*

Our main interest in this section is the ideal $\mathrm{fin}\otimes\mathrm{fin}$ on $\omega\times\omega$, which is defined by

$$X\in\mathrm{fin}\otimes\mathrm{fin} \text{ iff } \{n<\omega \mid (X)_n \text{ is infinite}\} \text{ is finite.}$$

Proposition 2.2. $(\mathrm{fin}\otimes\mathrm{fin},\subseteq)\equiv_T\omega^\omega$, *where on* ω^ω *we consider the everywhere domination order.*

Proof. By Fact 1.1 and the previous fact, $(\mathrm{fin}\otimes\mathrm{fin},\subseteq)\equiv_T((\mathrm{fin}\otimes\mathrm{fin})^*,\supseteq)\equiv_T(\mathrm{fin}^*\cdot\mathrm{fin}^*,\supseteq)$. By Proposition 1.6,

$$(\mathrm{fin}^*\cdot\mathrm{fin}^*,\supseteq)\equiv_T\prod_{n<\omega}\mathrm{fin}^*\equiv_T\prod_{n<\omega}\mathrm{fin}\equiv_T\prod_{n<\omega}\omega=\omega^\omega.$$

□

Corollary 2.3. *Suppose that* U *is an ultrafilter over* $\omega\times\omega$ *such that* $\mathrm{fin}\otimes\mathrm{fin}\subseteq U^*$, $\mathrm{fin}\otimes\mathrm{fin}$-*p.i.p. holds for* U, *and* $\prod_{n<\omega}(\omega^\omega,\leq)\leq_T U$. *Then* $U\cdot U\equiv_T U$.

Proof. Apply Corollary 1.17 for $I=\mathrm{fin}\otimes\mathrm{fin}$. □

The order $\prod_{n<\omega}\omega^\omega$ can be simplified:

Fact 2.4. $\prod_{n<\omega}\omega^\omega$ *is order isomorphic to* ω^ω *and in particular* $\prod_{n<\omega}\omega^\omega\equiv_T\omega^\omega$.

Proof. Take any partition of ω into infinitely many infinite sets $\langle A_n\mid n<\omega\rangle$. Then any function $f:\omega\to\omega$ induces functions $\langle f\restriction A_n\mid n<\omega\rangle\in\prod_{n<\omega}\omega^{A_n}$. Clearly, ω^{A_n} is isomorphic to ω^ω by composing each function $f:A_n\to\omega$ with the inverse of the transitive collapse $\pi_n:A_n\to\omega$. □

We now look for conditions that guarantee that $U\geq_T\omega^\omega$. One way, is to ensure that the Rudin-Keisler projection on the first coordinate is rapid:

Definition 2.5. Suppose that U is an ultrafilter such that $\mathrm{fin}\otimes\mathrm{fin}\subseteq U^*$. We say that U is 2-*rapid* if the ultrafilter $\pi_*(U)=\{X\subseteq\omega\mid\pi^{-1}X\in U\}$ is a rapid ultrafilter on ω, where $\pi:\omega\times\omega\to\omega$ is the projection to the first coordinate.

Corollary 2.6. *Suppose that* U *is an ultrafilter on* $\omega\times\omega$ *such that* $\mathrm{fin}\otimes\mathrm{fin}\subseteq U^*$, *and* U *is* $\mathrm{fin}\otimes\mathrm{fin}$-*p.i.p. and* 2-*rapid. Then* $U\cdot U\equiv_T U$.

Given an ideal I on a set X, an I-*positive* set is any set in $I^+:=P(X)\setminus I$. The forcing $P(X)/I$ is forcing equivalent to $(I^+,\subseteq^I)$, where the pre-order is given by $X\subseteq^I Y$ iff $X\setminus Y\in I$. If $G\subseteq P(X)$ is I^+-generic over V, then G is an ultrafilter for the algebra $P^V(X)$ (namely $(V,\in,G)\models$ "G is an ultrafilter") and also $I\subseteq G^*$. We will only be interested in the case where X is a countable set and $P(X)/I$ is σ-closed. This is equivalent to the following property of I: we say that I is a σ-*ideal* if whenever $\langle A_n\mid n<\omega\rangle$ is a $\subseteq$-decreasing sequence of I-positive sets, there is an $A\in I^+$ such that for every $n<\omega$, $A\setminus A_n\in I$.

Given that $P(X)/I$ is σ-closed, the forcing does not add new reals. Hence, if G is I^+-generic over V then $P^V(X)=P^{V[G]}(X)$ and thus, G is an ultrafilter over X in $V[G]$. Clearly, fin is a σ-ideal, and it is well known that the generic ultrafilter for $P(\omega)/\mathrm{fin}$ is selective (and therefore a p-point and rapid):

Fact 2.7 (Folklore)**.** *If* G *is* $P(\omega)/\mathrm{fin}$-*generic over* V, *then* G *is a Ramsey ultrafilter in* $V[G]$.

In particular, G is a rapid p-point. By results in [13], $G \cdot G \equiv_T G < \mathcal{U}_{\text{top}}$, and in fact, by results in [26], G is Tukey-minimal among nonprincipal ultrafilters. However, this does not answer Question 0.1 as G is a p-point and therefore basically generated. We will answer Question 0.1 below.

Next, let us move to the forcing $P(\omega \times \omega)/\text{fin} \otimes \text{fin}$. This forcing was first considered in [29] and later in many papers including [17], [5], [8], [11], and [1]. Again, it is not hard to see that $\text{fin} \otimes \text{fin}$ is a σ-ideal. The following properties of the generic ultrafilter are due to Blass, Dobrinen and Raghavan [5] and Dobrinen [8]:

Theorem 2.8. *Let G be a $P(\omega \times \omega)/\text{fin} \otimes \text{fin}$-generic ultrafilter over V. Then:*

(1) *G is not Tukey top and is also not basically generated.*
(2) *$\pi_*(G)$ is $P(\omega)/\text{fin}$-generic over V, where $\pi : \omega \times \omega \to \omega$ is the projection to the first coordinate.*
(3) *G is the immediate successor of $\pi_*(G)$ in the Tukey order.*

Proof. For the convenience of the reader, we provide references to the proofs of the above:

(1) [5, Thms. 47 and 60].
(2) [5, Prop. 30]
(3) [8, Thm. 6.2].

□

Together with Theorem 2.8, the following theorem provides an answer to Question 0.1:

Theorem 2.9. *Let G be a $P(\omega \times \omega)/\text{fin} \otimes \text{fin}$-generic ultrafilter over V. Then*

(1) $\text{fin} \otimes \text{fin} \subseteq G^*$.
(2) *G satisfies $\text{fin} \otimes \text{fin}$-p.i.p.*
(3) *G is 2-rapid.*
(4) $G \cdot G \equiv_T G$.

Proof. (1) is trivial. To see (2), the argument is the same as showing that the forcing is σ-complete. For the self-inclusion of this paper, let us provide an indirect proof assuming σ-completeness. Let $\langle X_n \mid n < \omega \rangle \subseteq G$. Since the forcing is σ-complete, $\langle X_n \mid n < \omega \rangle \in V$. Let $X \in G$ be such that $X \Vdash \forall n, X_n \in \dot{G}$, then $X \leq X_n$. Otherwise, $X \setminus X_n \in (\text{fin} \otimes \text{fin})^+$ and then $(X \setminus X_n) \leq X$ is a stronger condition which forces that $X_n \notin \dot{G}$, contradiction. Hence for every $n < \omega$, $X \setminus X_n \in \text{fin} \otimes \text{fin}$.

For (3), we apply the previous theorem clause (4) to see that $\pi_*(G)$ is generic for $P(\omega)/\text{fin}$ and therefore rapid by Fact 2.7. Finally, (4) follows from $(1) - (3)$ and Corollary 2.6.

□

3. Transfinite iterates of fin

In this section we obtain analoguous results to the ones from the previous section, but for ultrafilters with higher cofinal-type complexity. To do so, we will consider the generic ultrafilters G_α obtained by the forcing $P(\omega^\alpha)/\text{fin}^{\otimes \alpha}$, where $1 \leq \alpha < \omega_1$ (see the paragraph following Theorem 3.1). Such ultrafilters were investigated in [8] and in yet unpublished work [6]. We point out that for $2 \leq \alpha$, G_α is not a p-point and not basically generated; and for $\beta < \alpha$, there are natural Rudin-Keisler projections from G_α to G_β.

Theorem 3.1 (Dobrinen)**.** *Suppose that* $1 \le \alpha < \omega_1$ *and* G_α *be a generic ultrafilter obtained by forcing with* $P(\omega^\alpha)/\mathrm{fin}^{\otimes\alpha}$ *over* V*. Then* G_α *is not Tukey top and also not basically generated. Moreover,*

(1) *For each* $1 \le k < \omega$*, the collection of Tukey types of ultrafilters Tukey-reducible to* G_k *forms a chain of length* k *consisting exactly of Tukey types of* G_n *for* $1 \le n \le k$*.* [8, Thm. 6.2]
(2) *For each* $\omega < \alpha < \omega_1$*, the Tukey types of the* G_β*,* $1 \le \beta \le \alpha$ *are all distinct and form a chain, but there are actually* 2^ω *many Tukey types below* G_α*.* [6]

The following recursive definition of $\mathrm{fin}^{\otimes\alpha}$, for $2 \le \alpha < \omega_1$, is well-known and has appeared in [1], [8], [9], and [20].

(1) At successor steps, $\mathrm{fin}^{\otimes\alpha+1} = \mathrm{fin} \otimes \mathrm{fin}^{\otimes\alpha}$ is the ideal on $\omega^{\alpha+1} = \omega \times \omega^\alpha$; explicitly, $A \subseteq \omega^{\alpha+1}$ is in $\mathrm{fin}^{\otimes\alpha+1}$ iff for all but finitely many n, $(A)_n \in \mathrm{fin}^{\otimes\alpha}$.
(2) For limit $\alpha < \omega_1$ we fix an increasing sequence $\langle \alpha_n \mid n < \omega\rangle$ with $\sup_{n<\omega} \alpha_n = \alpha$ and define $\mathrm{fin}^{\otimes\alpha} = \sum_{\mathrm{fin}} \mathrm{fin}^{\otimes\alpha_n}$ on $\omega^\alpha := \biguplus_{n<\omega}\{n\} \times \omega^{\alpha_n}$; explicitly, $A \subseteq \omega^\alpha$ is in $\mathrm{fin}^{\otimes\alpha}$ iff for all but finitely many n, $(A)_n$ is in $\mathrm{fin}^{\otimes\alpha_n}$.

The Rudin-Keisler order is defined as follows: Let I, J be ideals on X, Y respectively. We say that $I \le_{RK} J$ if there is a function $f : Y \to X$ such that $f_*(J) = I$, where

$$f_*(J) = \{A \subseteq X \mid \pi^{-1}[X] \in J\}.$$

It is well known that the Rudin-Keisler order implies the Tukey order.

Lemma 3.2 (Folklore)**.** *For* $1 \le \beta \le \alpha < \omega_1$*, we have* $(\mathrm{fin}^{\otimes\beta}, \subseteq) \le_{RK} (\mathrm{fin}^{\otimes\alpha}, \subseteq)$ *and therefore* $(\mathrm{fin}^{\otimes\beta}, \subseteq) \le_T (\mathrm{fin}^{\otimes\alpha}, \subseteq)$.

Proof. By induction on α. At successor steps, we define the projection to the second coordinate, Rudin-Keisler projects $I \otimes J$ onto J and therefore $\mathrm{fin}^{\otimes\alpha+1}$ onto $\mathrm{fin}^{\otimes\alpha}$. For limit α, suppose that for every $m \le n < \omega$, $\pi_{n,m} : \omega^{\alpha_n} \to \omega^{\alpha_m}$ is a Rudin-Keisler projection of fin^{α_n} onto fin^{α_m}. Fix any $N < \omega$ Let us define $f : \omega^\alpha \to \omega^{\alpha_N}$ by applying

$$f(\langle k, x\rangle) = \begin{cases} f_{k,N}(x) & k \ge N \\ a^* & k < N \end{cases}$$

where a^* is any fixed element of ω^{α_N}. Now if $Y \subseteq \omega^{\alpha_N}$, then $f^{-1}[Y] = \cup_{n\ge N}\{n\} \times f^{-1}_{n,N}[Y]$ and $f^{-1}_{n,N}[Y]$. If $Y \in \mathrm{fin}^{\alpha_N}$ then $f^{-1}_{n,N}[Y] \in \mathrm{fin}^{\times\alpha_n}$ and therefore $f^{-1}[Y] \in \mathrm{fin}^{\otimes\alpha}$. If $Y \notin \mathrm{fin}^{\otimes\alpha_N}$, then $f^{-1}_{n,N}[Y] \notin \mathrm{fin}^{\otimes\alpha_n}$ and therefore $f^{-1}[Y] \notin \mathrm{fin}^\alpha$. Since $\le_{RK}$ is transitive, we conclude thee lemma. □

There is a simple characterization of the Tukey type of $\mathrm{fin}^{\otimes\alpha}$ given in the following theorem:

Theorem 3.3. *For every* $1 < \alpha < \omega_1$*,* $(\mathrm{fin}^{\otimes\alpha}, \subseteq) \equiv_T \omega^\omega$.

Proof. By induction on α. For $\alpha = 2$, this is Proposition 1.6. For successor α, by Proposition 1.6,

$$\mathrm{fin}^{\otimes\alpha+1} = \mathrm{fin} \otimes \mathrm{fin}^{\otimes\alpha} \equiv_T \mathrm{fin} \times \prod_{n<\omega} \mathrm{fin}^{\otimes\alpha}.$$

By the induction hypothesis, $\mathrm{fin}^{\otimes\alpha} \equiv_T \omega^\omega$, and $\mathrm{fin} \equiv_T \omega$. Therefore, by Fact 2.4,

$$\mathrm{fin} \times \prod_{n<\omega} \mathrm{fin}^{\otimes\alpha} \equiv_T \omega \times \prod_{n<\omega} \omega^\omega \equiv_T \omega \times \omega^\omega \equiv_T \omega^\omega.$$

So we conclude that $\mathrm{fin}^{\otimes\alpha+1} \equiv_T \omega^\omega$. For limit α, we have by Theorem 1.5 that

$$\mathrm{fin}^{\otimes\alpha} = \sum_{\mathrm{fin}} \mathrm{fin}^{\otimes\alpha_n} \leq_T \omega \times \prod_{n<\omega} \omega^\omega \equiv_T \omega^\omega.$$

For the other direction, we have by the previous lemma that $\omega^\omega \equiv_T \mathrm{fin}^{\otimes 2} \leq_T \mathrm{fin}^{\otimes\alpha}$, as desired. □

Definition 3.4. We say that an ultrafilter U on ω^α is *α-rapid* if $\pi_*(U)$ is rapid. where π is the projection to the first coordinate.

It is clear that if U is α-rapid, then $\omega^\omega \leq_T \pi_*(U) \leq_{RK} U$; hence we have the following:

Corollary 3.5. *Suppose that U is an α-rapid ultrafilter over ω^α such that $\mathrm{fin}^{\otimes\alpha} \subseteq U^*$ and $\mathrm{fin}^{\otimes\alpha}$-p.i.p. holds. Then $U \cdot U \equiv_T U$.*

Proof. By Corollary 1.17 for $I = \mathrm{fin}^{\otimes\alpha}$, it remains to verify that $\prod_{n<\omega} \mathrm{fin}^{\otimes\alpha} \leq_T U$. Indeed, $\prod_{n<\omega} \mathrm{fin}^{\otimes\alpha} \equiv_T \prod_{n<\omega} \omega^\omega$ and therefore by Fact 2.4, $\prod_{n<\omega} \mathrm{fin}^{\otimes\alpha} \equiv_T \omega^\omega$. Since U is α-rapid, $\omega^\omega \leq_T \pi_*(U) \leq_{RK} U$ and therefore $\prod_{n<\omega} \mathrm{fin}^{\otimes\alpha} \leq_T U$. It follows that $U \cdot U \equiv_T U$. □

The following fact that each $\mathrm{fin}^{\otimes\alpha}$ is a σ-ideal is well-known (see [1], [8], [9], [20]), and included here for self-containment.

Proposition 3.6. *Suppose that $\langle A_i \mid i < \omega\rangle$ is a decreasing sequence of sets in $(\mathrm{fin}^{\otimes\alpha})^+$. Then there is an $A \in (\mathrm{fin}^{\otimes\alpha})^+$ such that for every $i < \omega$, $A \setminus A_i \in \mathrm{fin}^{\otimes\alpha}$.*

Proof. By induction on α. For $\alpha = 1$, fin is indeed a σ-ideal. Suppose that $\mathrm{fin}^{\otimes\alpha}$ has proven to be a σ-ideal, and let $\langle A_i \mid i < \omega\rangle \subseteq (\mathrm{fin}^{\otimes\alpha+1})^+$ be a decreasing sequence. We may assume that each A_i is in standard form, namely, for every $n < \omega$, either $(A_i)_n = \emptyset$ or $(A_i)_n \in (\mathrm{fin}^{\otimes\alpha})^+$. Let

$$A = \bigcup_{i<\omega} \{(\pi'' A_i)(i)\} \times (A_i)_{(\pi'' A_i)(i)}$$

First we note that $A \in (\mathrm{fin}^{\otimes\alpha+1})^+$. To see this, note that since the $A_i's$ are decreasing then whenever $i < j$:

1. $\pi'' A_j \subseteq \pi'' A_i$.
2. For each $n < \omega$, $(A_j)_n \subseteq (A_i)_n$.

It follows that for $i < j < \omega$, $(\pi'' A_j)(j) \in \pi'' A_i$ and $(\pi'' A)(j) = (\pi'' A_j)(j) > (\pi'' A_i)(i) = (\pi'' A)(i)$. So $\{(\pi'' A_i)(i) \mid i < \omega\} \in \mathrm{fin}^+$ and for each i, $(A)_{(\pi'' A)(i)} = (\pi'' A_i)_{(\pi'' A_i)(i)} \in (\mathrm{fin}^{\otimes\alpha})^+$. To see that $A \setminus A_i \in \mathrm{fin}^{\otimes\alpha}$, for each $i \leq j$,

$$(A)_{(\pi'' A_j)(j)} (A_j)_{(\pi'' A_j)(j)} \subseteq (A_i)_{(\pi'' A_j)(j)}.$$

We conclude that $A \setminus A_i \subseteq \cup_{j<i} (\{(\pi'' A_j)_j\} \times (A_j)_{(\pi'' A_j)(j)} \in \mathrm{fin}^{\otimes\alpha}$. At limit steps, δ then the proof is completely analogous. □

Corollary 3.7. *Let G be $P(\omega^\alpha)/\mathrm{fin}^{\otimes\alpha}$-generic over V. Then G satisfies $\mathrm{fin}^{\otimes\alpha}$-p.i.p.*

Lemma 3.8. *If G is $P(\omega^\alpha)/\text{fin}^{\otimes\alpha}$-generic over V, then G is α-rapid.*

Proof. Let $f : \omega \to \omega$ be any function in $V[G]$. By σ-closure of $P(\omega^\alpha)/\text{fin}^{\otimes\alpha}$, $f \in V$. We proceed by a density argument, let $X \in P(\omega^\alpha)/\text{fin}^{\otimes\alpha}$, shrink X to $X_1 \in P(\omega^\alpha)/\text{fin}^{\otimes\alpha}$ so that X_1 is in standard form. By definition of $(\text{fin}^{\otimes\alpha})^+$, $\pi''X_1$ is infinite and so we can shrink $\pi''X_1$ to Y_1, still infinite such that $Y_1(n) \geq f(n)$. Define $X_2 = \cup_{n\in Y_1}\{n\} \times (X_1)_n$. Since X_1 was in standard form, for each $n \in Y_1$, $(X_1)_n$ is positive, and so, $X_2 \in (\text{fin}^{\otimes\alpha})^+$. Note that $X_2 \subseteq X$ and $\pi''X_2 = Y_1$. By density there is $X \in G$ such that for every $n < \omega$, $(\pi''X)(n) \geq f(n)$ and therefore $\pi_*(G)$ is rapid, namely G is α-rapid. □

As corollary we obtain the following theorem:

Theorem 3.9. *Suppose that G is a $P(\omega^\alpha)/\text{fin}^{\otimes\alpha}$-generic ultrafilter over V. Then $G \cdot G \equiv_T G$.*

Proof. We proved that $\text{fin}^{\otimes\alpha} \subseteq G^*$, G satisfies $\text{fin}^{\otimes\alpha}$-p.i.p. and that G is α-rapid. So by Corollary 3.5, $G \cdot G \equiv_T G$. □

Recalling Theorem 3.1, $G_\alpha <_T \mathcal{U}_{\text{top}}$ and G_α is not basically generated for each $\alpha < \omega_1$. The point is that although the complexity of the generic ultrafilter G_α increases with α, it still satisfies $G \cdot G \equiv_T G <_T \mathcal{U}_{\text{top}}$

4. Milliken-Taylor Ultrafilters

In this section, we prove that Milliken-Taylor ultrafilters have the same Tukey type as their Fubini product. Milliken-Taylor ultrafilters are ultrafilters on base set $\text{FIN} := [\omega]^{<\omega} \setminus \{\emptyset\}$ which witness instances of Hindman's Theorem [16]. They are the analogues of Ramsey ultrafilters on the base set FIN, but they are not Ramsey ultrafilters, nor even p-points, as shown by Blass in [4], where they were called *stable ordered union ultrafilters*. These ultrafilters have been widely investigated (see for instance [15] and [22]).

We now define Milliken-Taylor ultrafilters, using notation from [30]. For $n \leq \infty$, $\text{FIN}^{[n]}$ denotes the set of block sequences in FIN of length n, where a *block sequence* is a sequence $\langle x_i : i < n\rangle \subseteq \text{FIN}$ such $i < j < n$ implies $\max(x_i) < \min(x_j)$. For $n < \omega$ and a block sequence $X = \langle x_i : i < n\rangle \in \text{FIN}^{[n]}$, $[X] = \{\bigcup_{i\in I} x_i : I \subseteq n\}$. For an infinite block sequence $X = \langle x_i : i < \omega\rangle \in \text{FIN}^{[\infty]}$,

$$[X] = \{\bigcup_{i\in I} x_i : I \in \text{FIN}\}$$

For $X, Y \in \text{FIN}^{[\infty]}$, define $Y \leq X$ iff $[Y] \subseteq [X]$. Given $X \in \text{FIN}^{[\infty]}$ and $m \in \omega$, X/m denotes $\langle x_i : i \geq n\rangle$ where n is least such that $\min(x_n) > m$. Define $Y \leq^* X$ iff there is some m such that $[Y/m] \subseteq [X]$. Related definitions for finite block sequences are similar.

Definition 4.1. An ultrafilter U on base set FIN is *Milliken-Taylor* iff

1. For each $A \in U$, there is an infinite block sequence $X \in \text{FIN}^{[\infty]}$ such that $[X] \subseteq A$ and $[X] \in U$; and
2. For each sequence $\langle X_n : n < \omega\rangle$ of block sequences such that $X_0 \geq^* X_1 \geq^* \ldots$ and each $[X_n] \in U$, there is a diagonalization $Y \in \text{FIN}^{[\infty]}$ such that $[Y] \in U$ and $X_n \geq^* Y$ for each $n < \omega$.

Thus, a Milliken-Taylor ultrafilter U has $\{A \in U : \exists X \in \mathrm{FIN}^{[\infty]}\,(A = [X])\}$ as a filter base, and such a filter base has the property that almost decreasing sequences have diagonalizations. In this sense, Milliken-Taylor ultrafilter behave like p-points even though, technically, they are not. The following ideal corresponds to property (2):

Definition 4.2. Let I be the set of all $X \subseteq \mathrm{FIN}$ such that for some $N \in \omega$, $\forall x \in X, x \cap N \neq \emptyset$.

Claim 4.3. *I is an ideal and $I \subseteq U^*$ for each Milliken-Taylor ultrafilter U.*

Proof. Clearly, $\emptyset \in I$ and I is downwards closed. To see that I is closed under unions, let $X, Y \in I$ and let $N_X, N_Y \in \omega$ witness this. Then $\max(N_X, N_Y)$ witnesses that $X \cup Y \in I$. By condition (1), every $A \in U$ contains $[X]$ for some infinite block sequence X; in particular, $A \notin I$. □

Proposition 4.4. *If $Y \leq^* X$ then $[Y] \setminus [X] \in I$.*

Proof. If $Y \leq^* X$ then there is m such that $[Y/m] \subseteq [X]$ and so every element $b \in [Y] \setminus [X]$ must be a finite union of sets which includes some element below m. □

Proposition 4.5. *U satisfies I-p.i.p.*

Proof. Let $\langle A_n \mid n < \omega\rangle \subseteq U$. Then by property (1) of U, we can shrink each A_n to $[X_n] \in U$ such that $X_{n+1} \leq X_n$. By property (2) there is $[X] \in U$ such that $[X] \leq^* [X_n]$ for every n. Thus $[X] \setminus [X_n] \in I$ and in particular $[X] \setminus A_n \in I$. □

Proposition 4.6. *$I \equiv_T \omega$.*

Proof. Define $f : \omega \to I$ by $f(n) = \{x \in \mathrm{FIN} \mid x \cap n \neq \emptyset\}$. Clearly f is monotone, and by definition of I, f is cofinal. It is also clear that f is unbounded since $\cup_{n\in A} f(n) = \mathrm{FIN}$, whenever $A \in [\omega]^\omega$ (and I in a proper ideal). □

The projection maps min and max from FIN to ω are clear. Given a Milliken-Taylor ultrafilter U, let $U_{\min}$ and $U_{\max}$ denote the Rudin-Keisler projections of U according to min and max, respectively. Blass showed in [4] that $U_{\min}$ and $U_{\max}$ are both Ramsey ultrafilters. Hence, it follows that $\omega^\omega \leq_T U_{\min} \leq_{RK} U$, and therefore we have the following corollary:

Corollary 4.7. *$\prod_{n<\omega} I \leq_T U$.*

Theorem 4.8. *Suppose that U is a Milliken-Taylor ultrafilter. Then $U \cdot U \equiv_T U$.*

Proof. We proved that if U is Milliken-Taylor, then for the ideal I, we have that $I \subseteq U^*$, I-p.i.p., and $\prod_{n<\omega} I \leq_T U$. By Corollary 1.17, we conclude that $U \cdot U \equiv_T U$. □

We conclude this section with a short proof that the min-max projection of U is Tukey equivalent to its Fubini product with itself. The map min-max : $\mathrm{FIN} \to \omega \times \omega$ is defined by $\text{min-max}(x) = (\min(x), \max(x))$, for $x \in \mathrm{FIN}$. Let U be a Milliken-Taylor ultrafilter and let $U_{\min,\max}$ denote the ultrafilter on $\omega \times \omega$ which is the min-max Rudin-Kesiler projection of U. Blass showed in [4] that $U_{\min,\max}$ is isomorphic to $U_{\min} \cdot U_{\max}$ and hence, $U_{\min,\max}$ is not a p-point. Dobrinen and Todorcevic showed in [13] that $U_{\min,\max}$ is not a q-point, but is rapid, and that, assuming CH, $U_{\min}$ and $U_{\max}$ are Tukey strictly below $U_{\min,\max}$ which is Tukey strictly below U. It follows from the proof of Theorem 72 in [13] that $U_{\min,\max}$ has appropriately defined diagonalizations and hence, has the J-p.i.p. for the ideal $J = \{\text{min-max}(A) : A \in I\} \subseteq U^*_{\min,\max}$ on $\omega \times \omega$; hence the work in this paper implies the following theorem. However, we give a shorter proof by combining results from [4] and [13].

Corollary 4.9. *If U is a Milliken-Taylor ultrafilter, then*

$$U_{\min,\max} \equiv_T U_{\min,\max} \cdot U_{\min,\max}.$$

Proof. By results of Blass in [4], $U_{\min,\max} \cong U_{\min} \cdot U_{\max}$, and both $U_{\min}$ and $U_{\max}$ are Ramsey ultrafilters. For rapid p-points U, V, a result in [13] showed that $U \cdot V \equiv_T V \cdot U$, and hence,

$$(U \cdot V) \cdot (U \cdot V) \equiv_T U \cdot (V \cdot V) \cdot U \equiv_T U \cdot V \cdot U \equiv_T U \cdot U \cdot V \equiv_T U \cdot V.$$

The corollary follows. □

Remark 4.10. The theorems in this section should generalize to Milliken-Taylor ultrafilters on $\text{FIN}_k^{[\infty]}$ as well as their Rudin-Keisler projections, as their diagonalization properties will imply the I-p.i.p. for the naturally associated ideal I.

5. Further directions and open questions

Question 5.1. Is it a *ZFC* theorem that for any two ultrafilters U, V over ω, $U \cdot V \equiv_T V \cdot U$?

For κ-complete ultrafilters over measurable cardinals κ, this is indeed the case, as was proved by the authors in [2]. However, the proof essentially uses the well foundedness of the ultrapower by a κ-complete ultrafilter U.

A natural strategy to answer the previous question would be to take U such that $U <_T U \cdot U$. The only constructions of ultrafilters U such that $U <_T U \cdot U$ ensure that $U \not\geq_T \omega^\omega$. By the results of this paper we can generate examples where $U \not\geq_T \prod_{n<\omega} I$ for some ideal I such that U is I-p.i.p.

Using such U, we need to find an ultrafilter V such that $U \cdot V \not\equiv_T V \cdot U$. We know that following hold:

$$U \cdot V \equiv_T U \times V \cdot V, \quad V \cdot U \equiv_T V \times U \cdot U$$

So natural assumptions would be to require that $V \equiv_T V \cdot V$, and in order for V not to interfere with the assumption $U <_T U \cdot U$, in order to have $V \leq_T U$. This guarantees that

$$U \cdot V \equiv_T U <_T U \cdot U \equiv_T V \cdot U$$

However, the assumptions above are not consistent since if $V \cdot V \equiv_T V$ then $V \geq_T \omega^\omega$, and therefore if $V \leq_T U$ then also $U \geq_T \omega^\omega$. This leads to the following question:

Question 5.2. Is it consistent that there are two ultrafilters U, V such that $V \equiv_T V \cdot V \leq_T U <_T U \cdot U$? Or more precisely, is the class of ultrafilters which are Tukey reducible to their Fubini product upwards closed with respect to the Tukey order?

It seems that the Tukey type of ω^ω plays an important role in the calculations of the Tukey type of $U \cdot U$:

Question 5.3. Is there an ultrafilter U such that $\omega^\omega \leq_T U <_T U \cdot U$?

Question 5.4. Is it consistent to have an ultrafilter U such that U is not rapid but $U \geq_T \omega^\omega$? What about U which is a p-point?

Question 5.5. Is there a σ-ideal I on a countable set X such that some $P(X)/I$ generic ultrafilter U is Tukey-top?

Question 5.6. Is it true that for every σ-ideal I on a countable set X, a generic ultrafilter U on $P(X)/I$ satisfies $U \cdot U \equiv_T U$?

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