FORMS OF RAPIDNESS AND POWERS OF IDEALS

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ABSTRACT. In this paper we study ultrafilters which are Tukey above I^ω, where I is an ideal. In the first of the paper we use the I-p.i.p (pseudo intersection property) from [3] and deterministic ideals. Specifically, we prove the following two results for deterministic ideals:

(1) If U has the I-p.i.p, then $U \cdot U \equiv_T U \times I^\omega$, extending results from [24, 3].
(2) Ultrafilters without the I-p.i.p are always above I^ω.

Our main result involves a new hierarchy of ultrafilter– the α-almost rapid ultrafilters. We establish that the class of almost rapid ultrafilters is consistently strictly wider than the class of rapid ultrafilters, and give an example of a p-point ultrafilter which is almost rapid and non rapid. As a corollary, we obtain a p-point ultrafilter which is a non-rapid but is Tukey above ω^ω, answering [3, Q. 5.4].

0. INTRODUCTION

The Tukey order stands out as one of the most studied orders of ultrafilters [25, 14, 21, 10, 27, 2]. Its origins lie in the examination of Moore-Smith convergence, and it holds particular significance in unraveling the cofinal structure of the partial order $(U, \supseteq)$ of an ultrafilter. Formally, given two posets, $(P, \leq_P)$ and $(Q, \leq_Q)$ we say that $(P, \leq_P) \leq_T (Q, \leq_Q)$ if there is map $f : Q \to P$, which is cofinal, namely, $f''\mathcal{B}$ is cofinal in P whenever $\mathcal{B} \subseteq Q$ is cofinal. Schmidt [28] observed that this is equivalent to having a map $f : P \to Q$, which is unbounded, namely, $f''\mathcal{A}$ is unbounded in Q whenever $\mathcal{A} \subseteq P$ is unbounded in P. We say that P and Q are *Tukey equivalent*, and write $P \equiv_T Q$, if $P \leq_T Q$ and $Q \leq_T P$; the equivalence class $[P]_T$ is called the *Tukey type* or *cofinal type* of P.

A systematic study of the Tukey order on ultrafilter over ω, traces back to Isbell [18], later to Milovich [25] and Dobrinen and Todorcevic [14]. Lately, Benhamou and Dobrinen [2] extended this study to ultrafilters on cardinals greater than ω. Over measurable cardinals, the Tukey order is connected to recent developments revolving the so-called Galvin property, studied by Abraham, Garti, Goldberg, Gitik, Hayut, Magidor, Poveda, Shelah and others [1, 16, 15, 5, 9, 4, 7, 6, 17, 8]; the Galvin property in one of its forms is equivalent to being Tukey-top (i.e. Tukey maximal) as shown essentially by Isbell (in different terminology). Moreover, being Tukey-top in the restricted class of κ-complete ultrafilters takes the usual form of the Galvin property.

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In this paper, we study the class of ultrafilters U such that $(U, \supseteq) \geq_T (\omega^\omega, \leq)$, where $\leq$ is the *everywhere domination order* on ω^ω. This type of study has been considered before [31, 20, 21] in the context of the Tukey order on analytic ideals. In the context of general ultrafilters on ω, Louveau-Velickovic showed that ω^ω is immediate successor of the Tukey type ω [20]. More precisely, they show that if I is any ideal such that $I <_T \omega^\omega$ then I is countably generated. On the other hand, for analytic ideals p-ideal, Todorcevic [32] (See also [21, Thm 6.6]) showed that, they are either I countable generated or above ω^ω. This was later improved by Solecki and Todorcevic [31, Proposition 4.3] to show that if I is analytic, not locally compact ideal, then $I \geq_T \omega^\omega$. Later, Milovich asked [25, Question 4.7] if there is an ultrafilter U over ω such that $(U, \supseteq) \equiv_T \omega^\omega$. We will observe that this was basically answered in [31, Cor. 54] [1]:

Theorem 0.1 (Solecki-Todorcevic)**.** *Suppose that D is an ordered separable metric space such that the predecessors of each element form a compact set, and E is a basic [2] analytic order such that $D \leq_T E$, then D is analytic.*

Corollary 0.2. *There is no ultrafilter U over ω such that $(U, \supseteq) \equiv_T \omega^\omega$.*

Proof. By Sierpinski [30], a non-principal ultrafilter over ω is a non-measurable set as a subset of 2^ω and in particular non-analytic. An ultrafilter U with the topology inherited from 2^ω is a separable metric space and the set of $\supseteq$-predecessors is compact. Also, ω^ω is a basic analytic order, hence by Theorem 0.1, $(U, \supseteq) \not\leq_T \omega^\omega$. □

The following theorem contributed a great deal to the understanding of this class [14, Thm. 35]:

Theorem 0.3 (Dobrinen-Todorcevic)**.** *The following are equivalent for p-points:*[3]

(1) $U \cdot U \equiv_T U$.
(2) $U \geq_T \omega^\omega$.

Dobrinen and Todorcevic observed that rapid [4] ultrafilters are Tukey above ω^ω and deduced that rapid p-points satisfy $U \cdot U \equiv_T U$. Later, Milovich [24] gave a precise expression $U \cdot U$ which works for any p-point:

Theorem 0.4 (Milovich)**.** *If U is a p-point then $U \cdot U \equiv_T U \times \omega^\omega$.*

Recently, Benhamou and Dobrinenn [3] came back to the subject and worked under the general setup of the I-p.i.p (see Definition 1.3), which generalizes the notion of a p-point.

Theorem 0.5 (Benhamou-Dobrinen)**.** *Let U be an ultrafilter. Then the following are equivalent:*

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[1] Milovich's question appeared only 4 years after Solecki and Todorcevic's result.

[2] For the definition of basic see [31, §3].

[3] An ultrafilter U is a p-point if for every $\langle A_n \mid n < \omega \rangle \subseteq U$ there $X \in U$ such that for every $n < \omega$, $X \subseteq^* A_n$, i.e. $X \setminus A_n$ is finite.

[4] An ultrafilter U is rapid, if for every $f : \omega \to \omega$ there is $X \in U$ such that $f_X \geq f$, where $f_X(n)$ is the n-th element of X.

(1) $U \cdot U \equiv_T U$.
(2) *There is an ideal $I \subseteq U^*$ such that $U \geq_T I^\omega$ and U has the I-p.i.p.*

Taking $I = \text{fin}$ reproduces a part of Theorem 0.3. There is a slight difference between the type of equivalence to $U \cdot U \equiv_T U$ for p-points described in Theorem 0.3 and the one in 0.5. Indeed, in the latter, the ideal I can vary. If I is fixed, it is unclear whether for U which has the I-p.i.p, $U \cdot U \equiv_T U$ iff $U \geq_T I^\omega$. The reason that the ideal (i.e. fin) can be fixed in Theorem 0.3 is that every ultrafilter U which extends fin must also be Tukey above fin. This is what motivated Definition 3.1 of *deteministic ideals* (see Definition 3.1). In §1, we shows that this is indeed the missing ingredient, and both generalize 0.3 and slightly relaxed the assumption that U is a p-point.

Theorem. *If $I \subseteq U^*$ is deterministic then the following are equivalent:*

(1) $U <_T U \cdot U$ *and U has the I-p.i.p.*
(2) $U \not\geq_T I^\omega$

The significant part of the proof is to show that if I is a deterministic ideal, and $\text{fin} \subseteq I \subseteq U^*$, then every ultrafilter $U \not\geq_T I^\omega$ must have the I-p.i.p. This is closely related to [14, Question 42]. We also generalize Milovich's formula in Theorem 1.2 to the setup of a general deterministic ideal I.

Theorem. *Let I be deterministic. If U has the I-p.i.p then $U \cdot U \equiv_T U \times I^\omega$.*

Motivated by the above results, in § 2, we study the class of p-point which are above ω^ω. While it is consistent that there are p-points which are not above ω^ω, Dobrinen and Todorcevic observed that rapid p-points ultrafilters must be above ω^ω [14]. The main results of this paper concern a new class of ultrafilters– α-almost-rapid ultrafilters (Definition 2.3)– a weakening of rapidness.

Theorem. *Suppose that U is α-almost-rapid, then $U \geq_T \omega^\omega$.*

We then prove that the class of almost rapid ultrafilters is consistently a strict extension of the class of rapid ultrafilters, even among p-points.

Theorem. *Assume CH. Then there is a non-rapid almost-rapid p-point ultrafilter.*

In particular, this is the first example of a non-rapid p-points which is Tukey above ω^ω.

Finally, in § 3, we further study the I-p.i.p and deterministic ideal. In particular we show the following:

Theorem. *Suppose that U is not an accumulation point of $\mathcal{A} \subseteq \beta S$ (in the space of ultrafilters over S). Then U has that $(U \cap \bigcap \mathcal{A})^*$-p.i.p.*

The study of deterministic ideals yield the following theorem:

Theorem. *Suppose that I is a deterministic ideal and $U, U_0, U_1, \dots$ all the the I-p.i.p. Let $W = \sum_U U_n$, then W has the $I \cdot I$-p.i.p and in particular $W \cdot W \equiv_T W \times I^\omega$.*

For $I = \text{fin}$, the above theorem shows that Milovich's formula in 0.4 holds true sums of p-points as well.

Notations. $[X]^{<\lambda}$ denotes the set of all subsets of X of cardinality less than λ. Let $\mathrm{fin} = [\omega]^{<\omega}$, and $\mathrm{FIN} = \mathrm{fin} \setminus \{\emptyset\}$. For a collection of sets $(P_i)_{i\in I}$ we let $\prod_{i\in I} P_i = \{f : I \to \bigcup_{i\in I} P_i \mid \forall i,\ f(i) \in P_i\}$. If $P_i = P$ for every i, then $P^I = \prod_{i\in I} P$. Given a set $X \subseteq \omega$, such that $|X| = \alpha \leq \omega$, we denote by $\langle X(\beta) \mid \beta < \alpha\rangle$ be the increasing enumeration of X. Given a function $f : A \to B$, for $X \subseteq A$ we let $f''X = \{f(x) \mid x \in X\}$, for $Y \subseteq B$ we let $f^{-1}Y = \{x \in X \mid f(x) \in Y\}$, and let $\mathrm{rng}(f) = f''A$. Given sets $\{A_i \mid i \in I\}$ we denote by $\biguplus_{i\in I} A_i$ the union of the A_i's when the sets A_i are pairwise disjoint. Two partially ordered set $\mathbb{P}, \mathbb{Q}$ are isomorphic, denoted by $\mathbb{P} \simeq \mathbb{Q}$, if there is a bijection $f : \mathbb{P} \to \mathbb{Q}$ which is order-preserving.

1. Ultrafilters above I^ω

Given a set $\mathcal{F} \subseteq P(X)$, we denote by $\mathcal{F}^* = \{X \setminus A \mid A \in \mathcal{F}\}$. When $\mathcal{F}$ is a filter, $\mathcal{F}^*$ is an ideal which we call *the dual ideal*, and when $\mathcal{I}$ is an ideal $\mathcal{I}^*$ is a filter which we call *the dual filter*. Ideals are always considered with the (regular) inclusion order. For every filter F, $(F, \subseteq) \simeq (F^*, \supseteq)$ and in particular $(F, \subseteq) \equiv_T (F^*, \supseteq)$.

Recall that given a filter F (an ideal I) over X and filters $(F_x)_{x\in X}$ (ideals $(J_x)_{x\in X}$) over Y, the Fubini sum is a filter (ideal) over $X \times Y$, denoted $\sum_F F_x$ (denoted $\sum_I J_x$), and defined as follows: for every $A \subseteq X \times Y$

$$A \in \sum_F F_x \iff \left\{x \in X \mid (A)_x \in F_x\right\} \in F,$$

$$(A \in \sum_I J_x \text{ iff } \{x \in X \mid (A)_x \notin J_x\} \in I)$$

where $(A)_x = \{y \in Y \mid (x, y) \in A\}$. The Fubini product of T and S (ideals or filters) is obtained by setting $S_x = S$ for all $x \in X$, and $T \cdot S = \sum_T S$ When T is either an ideal of a filter on ω, define transifinitely for $\alpha < \omega_1$ the Fubini powers $T^{\oplus\alpha}$, by setting $T^{\otimes 1} = T$, at the successor step $T^{\otimes(\alpha+1)} = T^{\otimes\alpha} \cdot T$ and at limit steps, set $T^\alpha = \sum_T T^{\otimes\alpha_n}$, where $\langle\alpha_n \mid n < \omega\rangle$ is some fixed cofinal sequence in α.

Fact 1.1. $(\sum_I J_x)^* = \sum_{I^*} J_x^*$ and in particular $(I \cdot J)^* = I^* \cdot J^*$.

The following theorem provides the first step to analyze the Tukey type of a Fubini product of filters [24]:

Theorem 1.2 (Milovich)**.** *Let F, G be filters over ω, then $F \cdot G \equiv_T F \times G^\omega$ and in particular $F \cdot F \equiv_T F^\omega$.*

In follows inductively that for every $2 \leq \alpha < \omega_1$ $F^{\otimes\alpha} \equiv_T F \cdot F$. In [3], the following property was used to further invesigate this Tukey type of $F \cdot F$:

Definition 1.3. A filter F over a countable set S such that $I \subseteq F^*$ is an ideal, is said to satisfy the I-pseudo intersection property (I-p.i.p) if for every sequence $\langle X_n \mid n < \omega\rangle \subseteq F$, there is $X \in F$ such that for every n, $X \setminus X_n \in I$.

For example, being a p-point is equivalent to having the fin-p.i.p. More examples are obtained by considering ideals I such that $P(\omega)/I$ is σ-closed, then any generic ultrafilter U will satisfy the I-p.i.p (see [3]). The following proposition generalizes well-known characterizations of p-points (see e.g. [13]):

Proposition 1.4. *Let U be any ultrafilter over ω. Then the following are equivalent:*

(1) *U has the I-p.i.p.*
(2) *For any partition $\langle A_n \mid n<\omega\rangle$ such that for any n, $A_n\notin U$, there is $A\in U$ such that $A\cap A_n\in I$ for every $n<\omega$.*
(3) *Every function $f:\omega\to\omega$ which is unbounded modulo U is I-to-one modulo U, i.e. there is $A\in U$ such that for every $n<\omega$, $f^{-1}[n+1]\cap A\in I$.*

Proof. For (1) $\Rightarrow$ (2), let $\langle A_n \mid n<\omega\rangle$ be a partition such that $A_n\notin U$. Let $B_n=\omega\setminus A_n\in U$ and by the I-p.i.p there is $A\in U$ such that $A\setminus B_n\in I$. It remains to note that $A\setminus B_n=A\cap A_n$ to conclude (2).

To see (2) $\Rightarrow$ (3) let $f:\omega\to\omega$ be unbounded modulo U. Let $A_n=f^{-1}[\{n\}]$, then $A_n\notin U$. Apply (2) to the partition $\langle A_n\mid n<\omega\rangle$ to find $A\in U$ such that $A\cap A_n\in I$. For any $n<\omega$,

$$f^{-1}[n+1]\cap A=\cup_{m\leq n}f^{-1}[\{m\}]\cap A\in I.$$

Hence f is I-to-one modulo U.

Finally, to see (3) $\Rightarrow$ (1), let $\langle B_n\mid n<\omega\rangle\subseteq U$, and let us assume without loss of generality that it is $\subseteq$-decreasing and that $\bigcap_{n<\omega}B_n=\emptyset$. Define

$$f(n)=\min\{m\mid n\notin B_m\}.$$

Since $\bigcap_{n<\omega}B_n=\emptyset$, $f:\omega\to\omega$ is a well defined function. Apply (3), to find $A\in U$ such that for every $n<\omega$ $f^{-1}[n+1]\cap A\in I$. Now for each $x\in A\setminus B_n$, $f(x)\leq n$ and therefore $x\in f^{-1}[n+1]\cap A$ and therefore $A\setminus B_n\subseteq f^{-1}[n+1]\cap A\in I$. It follows that $A\setminus B_n\in I$ and that U has the I-p.i.p. □

The I-p.i.p was used to further analyze the Tukey type of $F\cdot F$:

Proposition 1.5 ([3])**.** *Suppose that F is a filter and $I\subseteq F^*$ is any ideal such that F has the I-p.i.p. Then $F^\omega\leq_T F\times I^\omega$.*

This was used to prove Theorem 0.5. Let us use it to prove the following Proposition which generalized 1.2 and 0.5

Proposition 1.6. *Let I be any ideal. Then for any ultrafilter $U\geq_T I$ which has the I-p.i.p, $U\cdot U\equiv_T U\times I^\omega$. Therefore, the following are equivalent for any ultrafilter $U\geq_T I$ which has the I-p.i.p:*

(1) *$U\cdot U\equiv_T U$.*
(2) *$U\geq_T I^\omega$.*

Proof. By Theorem 1.2, $U\cdot U\equiv_T U^\omega$. Since $I\leq_T U$, we have that $I^\omega\leq_T U^\omega$. Together with Proposition 1.5, we conclude that

$$U\times I^\omega\leq_T U^\omega\equiv_T U\cdot U\leq_T U\times I^\omega.$$

Now to see the equivalence, (2) $\Rightarrow$ (1) follows from Theorem 0.5, and (1) $\Rightarrow$ (2) follows from the first part as $U\cdot U\equiv_T U\times I^\omega\leq_T U\leq_T U\cdot U$. □

Definition 1.7. We say that an ideal I is *deterministic* if there is a cofinal set $\mathcal{B}\subseteq I$ such that for every $\mathcal{A}\subseteq\mathcal{B}$, $\bigcup\mathcal{A}\in I$ or $\bigcup\mathcal{A}\in I^*$.

As we already pointed out fin is deterministic. Other examples, including $\mathrm{fin}\times\mathrm{fin}$, can be found in Section 3.

Remark 1.8. It follows for example that in Proposition 1.6, if I is deterministic, then we can remove the assumption $U \geq_T I$. Indeed the assumption that U has the I-p.i.p ensures that $I \subseteq U^*$ and by Proposition 3.2, $I \leq_T U$.

Specific ultrafilters where the Proposition 1.6 turn out to be useful are ultrafilters which are generic for $P(X)/I$. In [3] it was proven that a generic ultrafilter for $P(X)/I$ where I is a P^+-ideal[5] satisfies the I-p.i.p. This, together with Proposition 1.6, given the following corollary which generalizes the results from [3] in our abstract settings.

Corollary 1.9. *Let I be a deterministic ideal over ω, and $1 \leq \alpha < \omega$ such that $P(\omega^\alpha)/I^{\otimes\alpha}$ does not add reals. Then:*

(1) *for every $2 \leq \alpha < \omega_1$, $\Vdash_{P(\omega)/I^{\otimes\alpha}}$ "$\dot{G}\cdot\dot{G} \equiv_T \dot{G}$".*
(2) *For $\alpha = 1$, $\Vdash_{P(\omega)/I}$ "$\dot{G}\cdot\dot{G} \equiv_T \dot{G}\times I^\omega$"*

Proof. To prove (1), let G be V-generic. As we pointed out, G has the $I^{\otimes\alpha}$-p.i.p. It remains to see that $(I^{\otimes\alpha})^\omega \leq_T G$. By Corollary 3.8 $I^{\otimes\alpha}$ is also deterministic. It is easy to see that being deterministic is absolute, hence $I^{\otimes\alpha}$ is deterministic in the extension $V[G]$ and moreover $I^{\otimes\alpha} \subseteq G^*$. Since $\alpha \geq 2$, $(I^{\otimes\alpha})^\omega \equiv_T I^{\otimes\alpha} \leq_T G$. Thus by Theorem 0.5 $G\cdot G \equiv_T G$. (2) follows from Proposition 1.6 as G has the I-p.i.p and $I \leq_T G$. □

Remark 1.10. Note that the previous corollary works also under the assumption $\Vdash_{P(\omega)/I^{\otimes\alpha}}$ "$I^{\otimes\alpha} \leq_T \dot{G}$" instead of I being deterministic.

These results motive the study of the class of ultrafilters which are Tukey above I^ω, with a specific emphasis on deterministic ideals. The following theorem shows that for deterministic I's this class extends the class of ultrafilters which do not have the I-p.i.p.

Theorem 1.11. *Suppose that I is deterministic ideal, and $\mathrm{fin} \subseteq I \subseteq U^*$. Then if $U \not\geq_T I^\omega$, then U has the I-p.i.p*

Proof. Let us verify the equivalent condition (2) in Proposition 1.4. Let $\langle A_n \mid n < \omega\rangle$ be a partition of ω such thatcfor every n, $A_n \notin U$. We need to find $X \in U$ such that $X \cap A_n \in I$ for every n. Without loss of generality, suppose that $A_n \in I^+$ for every n. Since $\mathrm{fin} \subseteq I$, A_n is infinite and we can find a bijection $\pi : \omega \leftrightarrow \omega\times\omega$ such that $\pi'' A_i = \{i\}\times\omega$. Let $W = \pi_*(U)$ be the Rudin-Keisler isomorphic copy of U. For each $n < \omega$, consider the ideal $I_n = \pi_*(I \cap P(A_n))$ on $\{n\}\times\omega$. By Proposition 3.5(2), $I \cap P(A_n)$ is a deterministic and since $\pi \restriction A_n$ is one-to-one $I_n = \pi_*(I \cap P(A_n))$ is deterministic by Proposition 3.5(1) and $I_n \equiv_T I \cap P(A_n)$ by Proposition 3.5(3). By Theorem 3.6, it follows that $\sum_{\mathrm{fin}} I_n$ is deterministic. It is not hard to check that since I is deterministic, $I \equiv_T I \cap P(A_n) \equiv_T I_n$ and therefore

$$I^\omega \equiv_T \prod_{n<\omega} I_n \equiv_T \sum_{\mathrm{fin}} I_n$$

[5] I is called a P^+-deal if $P(X)/I$ is a σ-closed forcing.

Since $U \not\geq_T I^\omega$, $W \not\geq_T \sum_{\text{fin}} I_n$. Since $\sum_{\text{fin}} I_n$ is deterministic, it follows that $\sum_{\text{fin}} I_n \nsubseteq W^*$. Thus, there is $X' \in \sum_{\text{fin}} I_n \cap W$. Namely, for all but finitely many n's, $(X')_n \in I_n$. Since each $A_i \notin U$, we may assume that, $(X')_n \in I_n$. Let $X = \pi^{-1}[X']$, then for every $n < \omega$, $X \cap A_n \in I$ as $\pi'' X \cap A_n = \{x\} \cap (X)_n \in I_n$. $\square$

The proof of the above achieves a bit more, it shows that if U does not have the I-p.i.p for a deterministic ideal I, then I^ω is realized as a deterministic sub ideal of U. Taking $I = \text{fin}$ in the above we obtain the following corollary

Corollary 1.12. *Suppose that U is a non-principal ultrafilter such that $U \not\geq_T \omega^\omega$ then U is a p-point.*

As a corollary, we see that Proposition 1.6 and therefore also Theorem 0.3 can be slightly improved.

Corollary 1.13. *If $I \subseteq U^*$ is deterministic then the following are equivalent:*

(1) $U <_T U \cdot U$ *and* U *has the* I*-p.i.p.*
(2) $U \not\geq_T I^\omega$

2. Almost Rapid Ultrafilters

In this section we restrict our attention to ω^ω. Our goal is to study the class of ultrafilters which are Tukey above ω^ω. As observed by Dobrinen and Todorcevic, rapid ultrafilters form a subclass of those. Clearly, rapid ultrafilters do not characterize the class of ω^ω, since for example there could be no rapid ultrafilters at all [23], while there are always ultrafilters which are above ω^ω (e.g namely Tukey-top). Moreover, counter example exists in ZFC. To see this, let us use the following result of Miller [23, Thm. 4]:

Proposition 2.1 (Miller)**.** *For any two ultrafilters U, V on ω, $U \cdot V$ is rapid iff V is rapid.*

Now, Choquet [12] showed there is always non-rapid ultrafilter U. For any such U, the ultrafilter $U \cdot U$ is non-rapid and certainly above ω^ω.

Note however that this construction does not yield a p-point, and indeed, by Corollary 1.12 any non-rapid non p-point will do the job. Hence, the interesting examples (and the ones which are motivated by the results in the previous section) lay inside the class of p-point (see Figure 1).

Of course, by Shelah [29], it is possible that there are no p-points (see also Chodounský and Guzmán [11]). In which case, Corollary 1.12 yield that every ultrafilter is above ω^ω.

By yet another result of Shelah, in the Miller model [22], which is obtained by countable support iteration of the superperfect tree forcing of length ω_2 over a model of CH, every p-point is generated by $\aleph_1$-many sets. It is known that $\mathfrak{d} = \mathfrak{c}$ holds in that model. Therefore, every p-point is generated by less than $\mathfrak{d}$-many sets and in particular cannot be above ω^ω. Hence the class of p-points coincide with the class of ultrafilters which are not Tukey above ω^ω, and the dashed area in Figure 1 is void. The purpose of this section is to address the question raised in [3] whether rapid p-point are exactly those p-points which are above ω^ω. In other words, are

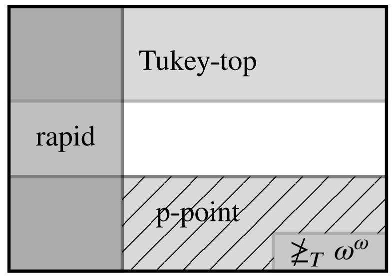

FIGURE 1. p-points but not $\geq_T \omega^\omega$.

there any other ways of producing cofinal maps from an ultrafilter to ω^ω? Let us introduce a new class of ultrafilters– the α-almost rapid ultrafilters– which does that.

Given a function $f : \omega \to \omega\setminus\{0\}$ such that $f(0) > 0$. We denote by $exp(f)(0) = f(0)$ and

$$exp(f)\big(n+1\big) = f\big(exp(f)(n)\big) = f(f(f(f...f(0)..))).$$

We define the n^{th} f-exponent function,

$$exp_0(f) = f \text{ and } exp_n(f) = exp(exp_{n-1}(f)).$$

Continuing transfinitely, for every $\alpha < \omega_1$:

$$exp_{\alpha+1}(f) = exp(exp_\alpha(f)).$$

For limit $\delta < \omega_1$, we fix some increasing cofinal sequence $\langle \delta_n \mid n < \omega\rangle$ in δ, and let

$$exp_\delta(f)(n) = \max\{exp_{\delta_n}(f)(n), exp_\delta(f)(n-1)+1\}.$$

Lemma 2.2. *Let $f, g : \omega \to \omega$ be increasing functions.*

(1) *For every $\alpha < \omega_1$, $exp_\alpha(f)$ is increasing.*
(2) *If $f \leq g$ then for every $\alpha < \omega_1$, $exp_\alpha(f) \leq exp_\alpha(g)$.*
(3) *For every $\alpha < \beta < \omega_1$, $exp_\alpha(f) <^* exp_\beta(f)$.*

Proof. For (1), we proceed by induction. For $\alpha = 0$, $exp_0(f) = f$ is increasing. Suppose $exp_\alpha(f)$ is increasing, then for every $n < \omega$, $exp_\alpha(f)(n) > n$. For $\alpha+1$, let $n < \omega$. Since $exp_\alpha(f)$ is increasing,

$$exp_{\alpha+1}(f)(n+1) = exp_\alpha(f)(exp_{\alpha+1}(f)(n)) > exp_{\alpha+1}(f)(n).$$

For limit δ, is clear from the definition that $exp_\delta(f)$ is increasing. Also (2) is proven by induction. The base case is $exp_0(f) = f \leq g = exp_0(g)$. Suppose this was true for α, and let us prove the induction step by induction on $n < \omega$. The base again is

$$\begin{aligned} exp_{\alpha+1}(f)(0) &= exp_\alpha(f)(0) \\ &\leq exp_\alpha(g)(0) \leq exp_{\alpha+1}(g)(0) \end{aligned}$$

Suppose that $exp_{\alpha+1}(f)(n) \leq exp_{\alpha+1}(g)(n)$, then by (1) and the induction hypothesis

$$\begin{aligned} exp_{\alpha+1}(f)(n+1) &= exp_\alpha(f)(exp_{\alpha+1}(f)(n)) \\ &\leq exp_\alpha(f)(exp_{\alpha+1}(g)(n)) \\ &\leq exp_\alpha(g)(exp_{\alpha+1}(g)(n)) = exp_{\alpha+1}(g)(n+1) \end{aligned}$$

At limit stages δ, by the induction hypothesis,

$$\begin{aligned} exp_\delta(f)(n) &= \max\{exp_{\delta_n}(f)(n), exp_\delta(f)(n-1)+1\} \\ &\leq \max\{exp_{\delta_n}(g)(n), exp_\delta(g)(n-1)+1\} = exp_\delta(g)(n). \end{aligned}$$

The proof of (3) is similar. □

Definition 2.3. For $\alpha < \omega_1$, we say that an ultrafilter U is *α-almost-rapid* if for every function $f \in \omega^\omega$ there is $X \in U$ such that $exp_\alpha(f_X) \geq^* f$, where f_X is the increasing enumeration of X.

Remark 2.4. By strengthening the above definition, we may require that $exp_\alpha(f_X) \geq f$. However, this strengthening yield an equivalent definition.

Note that 0-almost-rapid is rapid and if $\beta \leq \alpha$ then β-almost-rapid implies α-almost-rapid. We call U *almost-rapid* if it is 1-almost-rapid.

Proposition 2.5. *If U is α-almost-rapid implies $U \geq_T \omega^\omega$*

Proof. Consider the map $X \mapsto exp_\alpha(f_X)$. We claim that it is monotone and cofinal. First, suppose that $X \subseteq Y$, then the natural enumerations f_X, f_Y of X, Y (resp.) satisfy $f_X \geq f_Y$. Then by Lemma 2.2(3) $exp_\alpha(f_X) \geq exp_\alpha(f_Y)$. The map above is cofinal by the α-almost rapidness of U. □

The proposition below provides an analogous characterization to [23, Thm. 3] for almost-rapid ultrafilters.

Proposition 2.6. *The following are equivalent:*

(1) *U is almost-rapid.*
(2) *For any sequence $\langle P_n \mid n < \omega\rangle$ of sets, such that P_n is finite, there is $X \in U$ such that for each $n < \omega$, $exp(f_X)(n-1) \geq |X \cap P_n|$ (where $exp(f_X)(-1) = 0$).*
(3) *There is a function $h : \omega \to \omega$ such that for any $\langle P_n \mid n < \omega\rangle$ where P_n is finite, there is $X \in U$ such that for each $n < \omega$, $exp(f_X)(h(n-1)) \geq |X \cap P_n|$.*

Proof.

(1) ⇒ (2) Suppose that U is almost rapid, and let $\langle P_n \mid n < \omega\rangle$ be a sequence of finite subsets of ω. Let $f(n) = \max(P_n) + 1$. By (1), there is $X \in U$ such that $exp(f_X) \geq f$. It follows that

$$exp(f_X)(0) = f_X(0) = \min(X) > f(0) > \max(P_0).$$

More generally,

$$f_X(exp(f_X)(n-1)) = exp(f_X)(n) > f(n) > \max(P_n)$$

and therefore $|X \cap P_n| \leq exp(f_X)(n-1)$.

(2) ⇒ (1) Let f be any function. Let $P_n = f(n)$. Then by (2), there is X such that for each n, $|X \cap f(n)| \leq exp(f_X)(n-1)$ and therefore

$$f(n) < f_X(exp(f_X)(n-1)) = exp(f_X)(n).$$

(2) ⇒ (3) trivial.

(3) ⇒ (1) Fix h as in (3). Let f be any function and assume that f is increasing. Find $n_0 < n_1 < ... < n_k < ...$ such that for every k, $h(k) < n_k$. Set $P_n = f(n_k)$. Then by (3), there is $X \in U$ such that for each k, $|X \cap P_k| \leq exp(f_X)(h(k-1))$. Take any $n_k \leq m < n_{k+1}$, then

$$\begin{aligned}|X \cap f(m+1)| &\leq |X \cap f(n_{k+1})| \\ &\leq exp(f_X)(h(k)) \\ &\leq exp(f_X)(n_k) \leq exp(f_X)(m).\end{aligned}$$

Therefore $f(m+1) < f_X(exp(f_X)(m)) = exp(f_X)(m+1)$.

□

The following theorem is the main result of this paper:

Theorem 2.7. *Assume CH. Then there is a p-point which is almost-rapid but not rapid*

Proof. Let $P_n = \{1, ..., 2^n\}$. Let $I = \{A \subseteq \omega \mid \exists k \forall n > 0,\ |A \cap P_n| \leq k \cdot n\}$. Then I is a proper ideal on ω. Suppose that $\langle P_n \mid n < \omega\rangle$ is not a counterexample for U being rapid, then there is a set $X \in U$ such that $|X \cap P_n| \leq n$ for every n and therefore $X \in I$. Hence, as long as we have $U \subseteq I^+$, U will not be rapid.

Claim 2.8. *The following are equivalent:*

1. $A \in I^+$.
2. *for every k, there is n_k such that $|A \cap P_{n_k}| > k \cdot n_k$.*
3. *for every k, there are infinitely many n_k such that $|A \cap P_{n_k}| > k \cdot n_k$.*

Proof. (3) ⇒ (2) ⇒ (1) are trivial. Suppose that (3) fails, namely there is k_0 and n_0 such that for every $n \geq n_0$, $|A \cap P_n| \leq k_0 \cdot n$. Set $k_1 = \max\{k_0, |A \cap P_1|, ...|A \cap P_n|\}$, then for every $n > 0$, $|A \cap P_n| \leq k_1 \cdot n$ which implies that $A \in I$. Hence (1) ⇒ (3). □

The following is the key lemma for our construction:

Lemma 2.9. *Suppose that $\langle A_n \mid n < \omega\rangle \subseteq I^+$ is $\subseteq$-decreasing, and $f : \omega \to \omega$. Then there is $B \subseteq \omega$ such that*

1. $B \subseteq^* A_n$ *for every n.*
2. $B \in I^+$.
3. $exp(f_B) > f$.

Proof. Suppose without loss of generality that f is increasing. In particular, $f(k) \geq k$. Consider $f(1)$, find $2 < n_1$ so that

$$|A_1 \cap P_{n_1}| > (f(1) + 2) \cdot n_1$$

such an n_1 exists as $A_1 \in I^+$ and taking $k = f(1) + 2$ in 2.8(3). Find $a_0, ..., a_{n_1+1}$ such that:

(1) $f(0), n_1 + 1 < a_0 < a_1 < .... < a_{n_1+1}$.
(2) $a_{n_1+1} > f(1)$.
(3) $a_0, a_1, ..., a_{n_1+1} \in A_1 \cap P_{n_1}$.

It is possible to find such elements as

$$\begin{aligned}|A_1 \cap P_{n_1} \setminus \{0, ..., n_1 + 1\}| &> (f(1) + 2)n_1 - (n_1 + 2)\\ &\geq 3n_1 - n_1 - 2 = 2n_1 - 2 \geq n_1 + 1.\end{aligned}$$

So there are $n_1 + 1$ elements in $A_1 \cap P_{n_1}$ greater than $n_1 + 1$. Since $|A_1 \cap P_{n_1}| > f(1)$, we can also make sure that the $n_1 + 1$ element we choose is above $f(1)$. This way, we have arranged that:

(1) $f(0) < a_0$.
(2) a_{a_0} was not defined yet (!), but as long as the sequence is increasing, then $a_{a_0} > a_{n_1+1} > f(1)$.
(3) If B includes $a_0, ..., a_{n_1+1}$, then for $k = 1$, there is n_1 such that $|B \cap P_{n_1}| > n_1$.

Now consider $f(2)$ and by 2.8(3) find $n_2 > 2, a_{n_1+1}$ so that

$$|A_2 \cap P_{n_2}| > (f(2) + 2)(a_{n_1+1} + 1)n_2.$$

Hence we can choose $a_{n_1+2}, ..., a_{n_1+1+2n_2+1}$ such that

(1) $(n_1 + 1) + (2n_2 + 1) < a_{n_1+2} < ... < a_{n_1+1+2n_2+1}$.
(2) $f(2) < a_{n_1+1+2n_2+1}$.
(3) $a_{n_1+2}, ..., a_{n_1+1+2n_2+1} \in A_2 \cap P_2$.

This is possible to do since

$$\begin{aligned}|A_2 \cap P_{n_2} \setminus \{0, ..., n_1 + 2n_2 + 2\}| &> (f(2) + 2)(a_{n_1+1} + 2)n_2 - (n_1 + 2n_2 + 3)\\ &> 8n_2 - (3n_2 + 3)\\ &= 5n_2 - 3 > 2n_2 + 1.\end{aligned}$$

So we can find $a_{n_1+2}, ..., a_{n_1+1+2n_2+1}$ above $n_1 + 1 + 2n_2 + 1$ (and therefore also above a_{n_1+1}). We can also make sure that the last element we pick is above $f(2)$. This way we ensured the following:

(1) As we observed, a_{a_0} was not defined in the first round (and might not be defined in the second round as well) and therefore $a'_1 := a_{a_0} > n_1 + 1 + 2n_2 + 1$.
(2) $a_{a_{a_0}}$ was not defined but as long as the a_i's are increasing, $a'_2 := a_{a_{a_0}} > a_{n_1+1+2n_2+1} > f(2)$.
(3) For $k = 2$, there is n_2 such that $|B \cap P_{n_2}| > 2n_2$.

In general suppose we have defined $n_1 < n_2 < .... < n_k$ and $a_0, ..., a_{\sum_{i=1}^k in_i+1}$, such that $a'_{k-1} > \sum_{i=1}^k in_i + 1$. Then we find $n_{k+1} > k + 1, a_{\sum_{i=1}^k in_i+1}$ such that $|A_{k+1} \cap P_{n_{k+1}}| > 3(k + 1)(f(k + 1) + 1)n_{k+1}$. We now define

$$a_{(\sum_{i=1}^k in_i+1)+1}, a_{(\sum_{i=1}^k in_i+1)+2} ..., a_{\sum_{i=1}^{k+1} in_i+1}$$

(that is $(k+1)n_{k+1}+1$ many elements) so that:

(1) $\sum_{i=1}^{k+1} in_i + 1 < a_{(\sum_{i=1}^{k} in_i+1)+1} < ... < a_{\sum_{i=1}^{k+1} in_i+1}$,
(2) $a_{\sum_{i=1}^{k+1} in_i+1} > f(k+1)$.
(3) $a_{(\sum_{i=1}^{k} in_i+1)+1}, ..., a_{\sum_{i=1}^{k+1} in_i+1} \in A_{k+1} \cap P_{n_{k+1}}$.

To see that such a's exists, note that

$$|A_{k+1} \cap P_{n_{k+1}} \setminus \{0, ..., \sum_{i=1}^{k+1} in_i + 1\}|$$
$$> 3(k+1)(f(k+1)+1)n_{k+1} - (\sum_{i=1}^{k+1} in_i + 1) - 1$$
$$= 3(k+1)(f(k+1)+1)n_{k+1} - ((k+1)n_{k+1}+1) - (\sum_{i=1}^{k} in_i + 1) - 1$$
$$> (k+1)(3f(k+1)+3)n_{k+1} - 2((k+1)n_{k+1}+1)$$
$$= (k+1)(3f(k+1)+1)n_{k+1} > (k+1)n_{k+1}+1$$

Hence we can find $(k+1)n_{k+1}+1$-many elements in $A_{k+1}\cap P_{k+1}$ above $\sum_{i=1}^{k+1} in_i+1$. Also, since $|A\cap P_n| > f(k+1)$ we can make sure that $a_{\sum_{i=1}^{k+1} in_i+1} > f(k+1)$. This way we ensure that:

(1) Since $a_{a'_{k-1}}$ was not defined in previous rounds, and $a'_k := a_{a'_{k-1}} > \sum_{i=1}^{k+1} in_i + 1$.
(2) $a_{a'_k}$ has not been defined yet. Hence any future value for $a'_{k+1} := a_{a'_k}$ must satisfy that $a'_{k+1} > a_{\sum_{i=1}^{k+1} in_i+1} > f(k+1)f(k+1)$.
(3) $|B \cap P_{n_{k+1}}| > (k+1)n_{k+1}+1$.

Set $B = \{a_n \mid n < \omega\}$. So by the construction, for every k there is n_k such that $|B \cap P_{n_k}| > kn_k$. Hence $B \in I^+$. Also, note that $f_B(n) = a_n$ since the a_n's are increasing. By the construction and definition of $exp(f_B)$, $exp(f_B)(n) = a'_n > f(n)$. Finally, note that for each n, there is k such that for every $k' \geq k$, $a_{k'} \in A_m$ for some $m \geq n$. Since the sequence of A_n's is $\subseteq$-decreasing, $a_{k'} \in A_n$. We conclude that $B \setminus A_n \subseteq \{a_0, ..., a_k\}$. □

Back to the proof of Theorem 2.7, let us the construction a almost rapid p-point which is non-rapid. Enumerate $P(\omega) = \langle X_\alpha \mid \alpha < \omega_1\rangle$, and $P(\omega)^\omega = \langle \vec{A}_\alpha \mid \alpha < \omega_1\rangle$ such that each sequence in $P(\omega)^\omega$ appears cofinaly many times in the enumeration. Also enumerate $\omega^\omega = \langle \tau_\alpha \mid \alpha < \omega_1\rangle$. We define a sequence of filters V_α such that:

(1) $\beta < \alpha \Rightarrow V_\beta \subseteq V_\alpha$.
(2) $V_\alpha \subseteq I^+$.
(3) If $\alpha = \beta + 1$ then
    (a) either X_β or $\omega \setminus X_\beta \in V_\alpha$.
    (b) there is $X \in V_\alpha$ such that $\tau_\beta < exp(f_X)$.

(c) If $\vec{A}_\beta \subseteq V_\alpha$ then there is a pseudo-intersection $A \in V_\alpha$.

Let $V_0 = I^*$. At limit steps δ we define $V_\delta = \bigcup_{\beta<\delta} V_\beta$. It is clear that $(1)-(2)$ still holds at limit steps and (3) only concerns successor steps. At successors, given V_α, since we have only performed countably many steps so far, there are sets $B_n \in V_\alpha$ such that $V_\alpha = I^*[\langle B_n \mid n < \omega\rangle]$ where B_n is $\supseteq$-decreasing. If either X_α or $\omega \setminus X_\alpha$ is already in V_α, set $X^*_\alpha = \omega$. Otherwise, set $X_{\alpha^*} = X_\alpha$ and recall that $I^* = V_0 \subseteq V_\alpha$. It follows that $X_\alpha \in I^+$ and $V_\alpha[X_\alpha] \subseteq I^+$. Similarly, if $\vec{A}_\alpha \nsubseteq V_\alpha[X_\alpha]$ we set $\vec{A}^*_\alpha$ to have a trivial value such as $\langle \omega, \omega, \omega, ...\rangle$. Otherwise let $\vec{A}^*_\alpha = \vec{A}_\alpha$. Next, enumerate the set

$$\{B_n \cap X^*_\alpha \mid n < \omega\} \cup \{\vec{A}^*_\alpha(n) \mid n < \omega\} \subseteq V_\alpha[X_\alpha]$$

by $\langle B'_n \mid n < \omega\rangle$ and let $C_n = \cap_{m\leq n} B'_m$. By applying the previous lemma to the sequence $\langle C_n \mid n < \omega\rangle$, and τ_α, we can find $A^* \subseteq \omega$ such that:

(1) $A^* \in I^+$.
(2) $exp(f_{A^*}) > \tau_\alpha$.
(3) $A^* \subseteq^* C_n$ for every n.

Since for every $n < \omega$, there is n' such that $C_{n'} \subseteq \vec{A}_\alpha(n) \cap B_n$, $A^* \subseteq^* \vec{A}_\alpha(n)$, namely A^* is a pseudo intersection of both $\langle B_n \mid n < \omega\rangle$ and $\vec{A}_\alpha$. Also, A^* is a positive set with respect to the ideal $V_\alpha[X_\alpha]$. Otherwise, there is some $A \in I^*$ and B_n such that $A^* \cap (A \cap B_n \cap X_\alpha) = \emptyset$. But then $(A^* \cap B_n \cap X_\alpha) \cap A = \emptyset$ which implies that $A^* \cap B_n \cap X_\alpha \in I$. However, $A^* \subseteq^* B_n \cap X_\alpha$, which implies that $A^* \in I$, contradicting property (1) above in the choice of A^*. Hence we can define $V_{\alpha+1} = V_\alpha[X_\alpha, A^*]$ and $(1)-(5)$ hold.

This concludes the recursive definition. The ultrafilter witnessing the theorem is defined by $V^* = \bigcup_{\alpha<\omega_1} V_\alpha$.

Proposition 2.10. *V^* is a non-rapid almost-rapid p-point ultrafilter.*

Proof. V^* is an ultrafilter since for every $X \subseteq \omega$, there is α such that $X = X_\alpha$ and so either X_α or $\omega \setminus X_\alpha$ are in $V_{\alpha+1} \subseteq V^*$. Also V^* is a p-point since if $\langle A_n \mid n < \omega\rangle \subseteq V^*$ then there is $\alpha < \omega_1$ such that $\langle A_n \mid n < \omega\rangle \subseteq V_\alpha$ and by the properties of the enumeration there is $\beta > \alpha$ such that $\vec{A}_\beta = \langle A_n \mid n < \omega\rangle$. This means that in $V_{\beta+1}$ there is a pseudo intersection for the A_n's. It is non-rapid as $V^* \subseteq I^+$ and, in fact, the sequence $P_n = \{1, ..., 2^n\}$ witnesses that it is non-rapid. Finally, it is almost rapid since for any function $\tau : \omega \to \omega$, there is α such that $\tau = \tau_\alpha$ and therefore in $V_{\alpha+1}$ there is a set X such that $exp(f_X) > \tau$. □

□

Corollary 2.11. *It is consistent that there is a p-point which is not rapid but still above ω^ω.*

Remark 2.12. CH is not necessary in order to obtain such an ultrafilter, since we can, for example, repeat a similar argument in the iteration of Mathias forcing after we forced the failure of CH and obtain such an ultrafilter. In fact, we conjecture that the construction of Ketonen [19] of a p-point from $\mathfrak{d} = \mathfrak{c}$ can be modified to get a non-rapid almost-rapid p-point.

3. More on the I-p.i.p and deterministic ideals

In this section, we study the I-p.i.p and deterministic ideals. As we have seen in previous sections, together, they provide an abstract framework in which one can analyze the connection between the Tukey type of Fubini products and subideals related to it. Many of our results in this section generalize to κ-filters (i.e. κ-complete filters over $\kappa \geq \omega$).

3.1. **Simple and deterministic ideals.**

Definition 3.1. Let I be an ideal. We say that I is:

(1) *simple* if for every ideal J, $I \subseteq J$, $I \leq_T J$.
(2) *deterministic* if there is a cofinal set $\mathcal{B} \subseteq I$ such that for every $\mathcal{A} \subseteq \mathcal{B}$, $\bigcup \mathcal{A} \in I$ or $\bigcup \mathcal{A} \in I^*$.

The basic connection between the two notion is:

Proposition 3.2. *If I is deterministic then I is simple.*

Proof. Let $I \subseteq J$ and let $\mathcal{B} \subseteq I$ be the cofinal set witnessing that I is deterministic. Let us prove that the identity function $id : \mathcal{B} \to J$ is unbounded. Suppose that $\mathcal{A} \subseteq \mathcal{B}$ is unbounded, then $\bigcup \mathcal{A} \notin I$, since otherwise, as $\mathcal{B}$ is cofinal in I, there would have been $b \in \mathcal{B}$ bounding $\mathcal{A}$. By definition of deterministic ideals, it follows that $\bigcup \mathcal{A} \in I^*$, and since $I \subseteq J$, $I^* \subseteq J^*$ hence $\bigcup \mathcal{A} \in J^*$. We conclude that $\bigcup \mathcal{A} \notin J$, namely, $\mathcal{A}$ is unbounded in J. Hence the identity function witnesses that $I \equiv_T \mathcal{B} \leq_T J$. □

Clearly any dual of an ultrafilter is deterministic. Also fin deterministic as witnessed by ω viewed as a cofinal subset of fin. It is easy to construct non-simple (hence non-deterministic) ideals:

Example 3.3. Suppose that $U \not\geq_T W$ are two ultrafilters. For example U could be a p-point and W a Tukey-top ultrafilter. Consider $(U \cap W)^* = I$, then I is not simple, since $U \cap W \subseteq U$ but $U \cap W \equiv_T U \times W \not\leq_T U$.

Example 3.4. On a regular uncountable cardinal κ, the dual to the non-stationary ideal– generated by the complements of closed and unbounded sets in the order topology of κ– is deterministic. Indeed, the intersection of closed sets is always closed. Hence if the intersection of clubs is not a club, it had to be bounded in κ, which is therefore non-stationary.

Proposition 3.5. *Suppose that $I \subseteq X$ is a deterministic ideal over X.*

(1) *If $\pi : X \to Y$ is injective on a set in I^*. Then $\pi_*(I) := \{a \mid \pi^{-1}[a] \in I\}$ is deterministic.*
(2) *If $A \subseteq X$, then $I \cap P(A)$ is deterministic.*
(3) *For any $A \in I^+$, we have $I \equiv_T I \cap P(A)$.*

Proof. Along this proof let us fix once $\mathcal{B} \subseteq I$ a witnessing cofinal set for I being deterministic.

To see (1), let

$$\mathcal{C} = \{(Y \setminus (\pi[X \setminus b])) \mid b \in \mathcal{B}\}.$$

Then $\mathcal{C}$ is a cofinal set in $\pi_*(I)$. We claim that $\mathcal{C}$ witnesses that $\pi_*(I)$ is deterministic. Let $\mathcal{A} \subseteq \mathcal{B}$ be such that $\bigcup_{a\in\mathcal{A}} Y \setminus (\pi[X \setminus a]) \notin \pi_*(I)$. Then $\pi^{-1}[\bigcup_{a\in\mathcal{A}} Y \setminus (\pi[X \setminus a])] \notin I$. Using the fact that π is $1-1$ it follows that $\pi^{-1}[\bigcup_{a\in\mathcal{A}} Y \setminus (\pi[X \setminus a])] = \bigcup \mathcal{A}$. Hence $\bigcup \mathcal{A} \notin I$, and since $\mathcal{A} \subseteq \mathcal{B}$, we conclude that

$$X \setminus \pi^{-1}[\bigcup_{a\in\mathcal{A}} Y \setminus (\pi[X \setminus a])] = X \setminus \bigcup \mathcal{A} \in I.$$

Namely, $\bigcup_{a\in\mathcal{A}} Y \setminus (\pi[X \setminus a]) \in \pi_*(I)^*$.

For (2), consider $\mathcal{C} = \{b \cap A \mid b \in \mathcal{B}\}$. Then $\mathcal{C}$ is easely seen to witness that $I \cap P(A)$ is deteministic.

For (3), let $f : \mathcal{B} \to I \cap P(A)$ be the map $f(b) = b \cap A$. This is clearly a monotone and cofinal map. We claim it is also unbounded, suppose that $\bigcup \mathcal{A} \notin I$, then $\bigcup \mathcal{A} \in I^*$, then $A \setminus \bigcup f(\mathcal{A}) = A \setminus \bigcup \mathcal{A} \in I \cap P(A)$. Hence f is unbounded. □

Note that (2) above can be vacuous if $A \in I$, since in that case $I \cap P(A)$ is not proper. So we should at least assume that $A \in I^+$. Generally speaking, it is unclear whether an ideal relative to a positive set has the same Tukey-type. However, as (3) shows, if the ideal is deterministic, the type does not change.

Theorem 3.6. *Suppose that* $\mathrm{fin} \subseteq I$ *be a deterministic ideal over* ω*, and* $\langle J_n \mid n < \omega\rangle$ *is a sequence of deterministic ideals over* ω *such that for every* $n < \omega$*,* $J_{n+1} \geq_T J_n$*. Then* $\sum_I J_n$ *is deterministic.*

Remark 3.7. There is a näive "proof" to try ans show that $\sum_I J_n$ is deterministic whenever I, J_n are deterministic. Take $\mathcal{B} \subseteq I$ and $\mathcal{B}_n \subseteq J_n$ be witnessing cofinal sets, then let $\mathcal{C}$ consist of sets of the form $(\bigcup_{n\in B}\{n\}\times\omega)\cup(\bigcup_{n\notin B}\{n\}\times B_n)$, where $B \in \mathcal{B}$ and $B_n \in \mathcal{B}_n$. While $\mathcal{C}$ is cofinal in $\sum_I J_n$, it does not have the desired property. For example if $I = J_n = \mathrm{fin}$ and $\mathcal{B} = \mathcal{B}_n = \omega$. Fix any infinite co infinite set A and consider $X_n = \bigcup_{n\in A}\{n\}\times n$. Note that X_n is of the form described above. Clearly $\bigcup_{n<\omega} X_n = A \times \omega \notin \mathrm{fin}\cdot\mathrm{fin}$, but also not in $(\mathrm{fin}\cdot\mathrm{fin})^*$. The proof how to correct this construction.

Proof. Let $\mathcal{B}_n \subseteq J_n$ witness that J_n is deterministic, and $\mathcal{D}$ witness that I is deterministic. Let $\langle f_{m,n} : J_n \to J_m \mid n \leq m < \omega\rangle$ be a sequence of unbounded maps. Denote by $\mathcal{B}$ the set of all sequences $\vec{b} = \langle b_n \mid n < \omega\rangle \in \prod_{n<\omega}\mathcal{B}_n$ such that for every $n < m < \omega$, $b_m \supseteq f_{m,n}(b_n)$. For $\vec{b} \in \mathcal{B}$ and $A \in \mathcal{D}$ defined

$$C_{A,\vec{b}} = \bigcup_{n\in A}\{n\}\times\omega \cup \bigcup_{n\notin A}\{n\}\times b_n.$$

Let us show that $\mathcal{C} = \{C_{A,\vec{b}} \mid A \in \mathcal{D}, \vec{b} \in \mathcal{B}\}$ is cofinal in $\sum_I J_n$. Let $Z \in \sum_I J_n$, and $A = \{n \mid (Z)_n \notin J_n\}$. By definition of $\sum_I J_n$, $A \in I$, so there is $D \in \mathcal{D}$ such that $A \subseteq D$. Construct (recursively) an increasing sequence $\langle b_n \mid n < \omega\rangle \in \prod_{n<\omega}\mathcal{B}_n$ such that for every $n \notin A$, $(Z)_n \subseteq b_n$ and for every $n < m < \omega$, $f_{m,n}(b_n) \subseteq b_m$. It follows that for every n, $(Z)_n \subseteq (C_{D,\vec{b}})_n$ and therefore $Z \subseteq C_{D,\vec{b}}$. Let us prove that C witnesses that $\sum_I J_n$ is deterministic. Suppose that $\bigcup_{i\in T} C_{X_i,\vec{b}^i} \notin \sum_I J_n$. Then

$$A := \{n<\omega \mid (\bigcup_{i\in T} C_{X_i,\vec{b}^i})_n \notin J_n\} \notin I.$$

Note that $(\bigcup_{i\in T} C_{X_i,\vec{b}^i})_n = \bigcup_{i\in T}(C_{X_i,\vec{b}^i})_n$. Let us split into cases: If $A\subseteq \bigcup_{i\in T} X_i$, then $\bigcup_{i\in T} X_i \notin I$ and since $\{X_i \mid i\in T\}\subseteq \mathcal{D}$, $\bigcup_{i\in T} X_i \in I^*$. Now for every $n\in \bigcup_{i\in T} X_i$, $\omega = \bigcup_{i\in T}(C_{X_i,\vec{b}^i})_n \in J^*_n$. We conclude that

$$\{n<\omega \mid (\bigcup_{i\in T} C_{X_i,\vec{b}^i})_n \in J^*_n\} \supseteq \bigcup_{i\in T} X_i \in I^*.$$

Hence $\bigcup_{i\in T} C_i \in (\sum_I J_n)^*$.

Otherwise, consider $n_0\in A$ such that for every $i\in T$, $n_0\notin X_i$. Then $(C_{X_i,\vec{b}^i})_{n_0} \in \mathcal{B}_{n_0}$ for every $i\in T$ and $\bigcup_{i\in T}(C_{X_i,\vec{b}^i})_{n_0} \notin J_{n_0}$. Since $\mathcal{B}_{n_0}$ witnesses that J_{n_0} is deterministic, $\bigcup_{i\in T}(C_{X_i,\vec{b}^i})_{n_0} \in J^*_{n_0}$. For every $n\geq n_0$, either there is $i\in T$ such that $n\in X_i$, and as we have seen, $\cup_{i\in T}(C_{X_i,\vec{b}^i})_n = \omega \in J^*_n$. Otherwise, for every $i\in T$, $(C_{X_i,\vec{b}^i})_n = b^i_n \in \mathcal{B}_n$, and by the assumption, $f_{n,n_0}(b^i_{n_0})\subseteq b^i_n$. Since $\bigcup_{i\in T} b^i_{n_0} \notin J_{n_0}$, and f_{n,n_0} is unbounded, $\bigcup_{i\in T} f_{n,n_0}((C_{X_i,\vec{b}^i})_n) \notin J_n$. Since $\mathcal{B}_n$ witnesses that J_n is deterministic, it follows that $\bigcup_{i\in T} f_{n,n_0}((C_{X_i,\vec{b}^i})_n) \in J^*_n$. We conclude that for every $n\geq n_0$, $\bigcup_{i\in T}(C_{X_i,\vec{b}^i})_n \in J^*_n$. Since $\{n_0, n_0+1,...\}\in I^*$, we have that $\bigcup_{i\in T} C_{X_i,\vec{b}^i} \in (\sum_I J_n)^*$ as wanted.

□

Corollary 3.8. *Suppose that I, J are deterministic ideals over ω. Then $I\cdot J$ is deterministic. Hence also for every $\alpha<\omega_1$, $I^{\otimes\alpha}$ is deterministic.*

3.2. **The I-pseudo intersection property.** Recall that given a filter F and an ideal $I\subseteq F^*$, F satisfy the I-p.i.p if for every $\langle X_n \mid n<\omega\rangle \subseteq F$ there is $A\in F$ such that for every n, $A\setminus X_n\in I$. This is a generalization of begin a p-point in the case $T=\text{fin}$. Other examples for ultrafilters U and ideals I such that U has the I-p.i.p can be found in [3] and will be provided in this subsection.

Given a set S, consider the space of filters over S, $\mathfrak{B}(S) = \{F \mid F \text{ is a filter over } S\}$ equipped with the topology generated by the sets $\mathcal{O}_A = \{F\in\mathfrak{B} \mid A\in F\}$, for $A\subseteq S$.

Given a filter F on S, we would like to study the following set:

$$\text{PIP}(F) = \{T\in\mathfrak{B}(S) \mid F \text{ has the } T^*\text{-p.i.p}\}$$

The following facts are also easy to verify:

Fact 3.9.

(1) $F\in \text{PIP}(F)$.
(2) $\text{PIP}(F)$ is upward closed with respect to $\subseteq$.
(3) $\text{PIP}(F)$ is closed under finite intersections.
(4) $\{S\}\in\text{PIP}(F)$ if and only if F is σ-complete.
(5) Let $f: S\to S'$ be a function. If $T\in\text{PIP}(F)$ then $f(T)\in\text{PIP}(f_*(F))$.

Given a set $\mathcal{B}$ with the finite intersection property, we let $F_{\mathcal{B}} = \{X \subseteq S \mid \exists I \in [\mathcal{B}]^{<\omega}, \bigcap I \subseteq X\}$ be the filter generated by $\mathcal{B}$. PIP(F) is the following ultrafilter-like property:

Proposition 3.10. *Suppose that $T \in$ PIP(F), and $\mathcal{B}$ is a base for T (i.e. cofinal in $(T, \supseteq)$). Suppose that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. Then either $F_{\mathcal{B}_1} \in$ PIP(F) or $F_{\mathcal{B}_2} \in$ PIP(F).*

Proof. Suppose neither $F_{\mathcal{B}_1}$-p.i.p nor $F_{\mathcal{B}_2}$-p.i.p hold for F. Then there are sequences $\langle A_n \mid n < \omega\rangle$ and $\langle B_n \mid n < \omega\rangle$ such that for every A, B there are n_A, m_B such that $A \setminus A_{n_A} \notin F_{\mathcal{B}_1}$ and $B \setminus B_{m_B} \notin F_{\mathcal{B}_2}$. Consider the sequence $\langle \bigcap_{k\leq n} A_k \cap \bigcap_{k\leq n} B_k \mid n < \omega\rangle$. Then there is $A \in F$ such that for every l $A \setminus \bigcap_{k\leq l} A_k \cap \bigcap_{k\leq l} B_k \in T$. For A, there are suitable n_A, m_A as above and fix $N = \max(n_A, m_A)$. Since Without loss of generality, $A \setminus \bigcap_{k\leq N} A_k \cap \bigcap_{k\leq N} B_k \in T^*$, and $\mathcal{B}$ is a generating set for T, there is $B \in \mathcal{B}$ such that $A \setminus \bigcap_{k\leq N} A_k \cap \bigcap_{k\leq N} B_k \subseteq B^c$. Without loss of generality, we may assume that $B \in \mathcal{B}_1$, in which case, we have $A \setminus A_n \subseteq A \setminus \bigcap_{k\leq N} A_k \cap \bigcap_{k\leq M} B_k \in F^*_{\mathcal{B}_1}$, contradicting the choice of n_A. □

This proposition already given many non-trivial examples of $T \in$ PIP(F). By definition, F is an accumulation point of $\mathcal{A} \subseteq \mathfrak{B}(S)$ if and only if $F \subseteq \bigcup(\mathcal{A} \setminus \{F\})$.

Theorem 3.11. *Suppose that F is not an accumulation point of $\mathcal{A} \subseteq \beta S$ (i.e. $\mathcal{A}$ consist of ultrafilters). Then $F \cap \bigcap \mathcal{A} \in$ PIP(F).*

Proof. For the contrapositive, suppose that F does not have the $(\bigcap \mathcal{A})^*$-p.i.p, and let $\langle X_n \mid n < \omega\rangle \subseteq F$ be a sequence witnessing this. Namely, for every $X \in F$, there is n such that $X \setminus X_n \notin T^*$ for some $T \in \mathcal{A}$. Since $X \setminus X_n \in F^*$, it follows $F \neq T$. Also, since T is an ultrafilter $X \setminus X_n \in T$, hence $X \in T$. It follows that $F \subseteq \bigcup \mathcal{A} \setminus \{F\}$, contradiction. □

The above apply to the following specific case. Recall that a set of ultrafilters $\mathcal{A} \subseteq \beta S$ is *discrete* if there are sets $A_U \in U$ for all $U \in \mathcal{A}$ which are pairwise disjoint. This is just equivalent to being a discrete set in the space $\mathfrak{B}(S)$. In particular, no point $U \in \mathcal{A}$ is in the closure of $\mathcal{A}$. Also note that every finite set of ultrafilters is discrete.

Corollary 3.12. *Suppose that $\mathcal{A}$ is discrete then each $U \in \mathcal{A}$ has the $\bigcap \mathcal{A} \in$ PIP(U).*

One way to obtain non-trivial sets T for which an ultrafilter U has the T-p.i.p is by intersecting U with other ultrafilters:

Corollary 3.13. *Suppose that $U_1, U_2, ...U_n$ are any ultrafilters, then for each $1 \leq i \leq n$, U_i have the $(U_1 \cap U_2 \cap ... \cap U_n)^*$-p.i.p.*

We can conclude for example that $U \cdot U \leq_T U \times (U \cap W)^\omega$ for any ultrafilter W. However, it is easy to see that $U \cap W \equiv_T U \times W$. Hence applying the I-p.i.p here will not yield interesting bounds in the Tukey order.

Our next result investigates how the I-p.i.p is preserved under sums of ideals and ultrafilters.

Proposition 3.14. *Let F, F_n be filters over countable sets. Suppose that $I \subseteq F^*$ and $J_n \subseteq F_n^*$ are ideals for every $n < \omega$. If F has I-p.i.p and for every $n < \omega$, F_n has J_n-p.i.p, then $\sum_F F_n$ has $\sum_I J_n$-p.i.p.*

Proof. Let $\langle A_n \mid n < \omega \rangle$ be a sequence in $\sum_F F_n$. For each n, let

$$X_n = \{m < \omega \mid (A_n)_m \in F_m\} \in F.$$

We find $X \in F$ such that for every $n < \omega$, $X \setminus X_n \in I$. For each $m \in X$, we consider $E_m = \{n < \omega \mid m \in X_n\}$. If E_m is finite, we let $Y_m = \bigcap_{n \in E_m} (A_n)_m \in F_m$ (if E_m is empty, we let $Y_m = \omega$). Otherwise, we find $Y_m \in F_m$ such that for all $n \in E_m$, $Y_m \setminus (A_n)_m \in J_m$. Let

$$A = \bigcup_{m \in X} \{m\} \times Y_m.$$

Clearly, $A \in \sum_F F_n$. Let $n < \omega$, we would like to show that $A \setminus A_n \in \sum_I J_n$. Let $m < \omega$ be such that $(A \setminus A_n)_m \notin J_m$. Since $(A \setminus A_n)_m = (A)_m \setminus (A_n)_m$, it follows that $n \in X$ (otherwise $(A)_m = \emptyset$) and $m \notin X_n$. Indeed, if $m \in X_n \cap X$ then $n \in E_m$ and by the choice of Y_m, $(A)_m \setminus (A_n)_m = Y_m \setminus (A_n)_m \in J_m$. We conclude that

$$\{m < \omega \mid (A \setminus A_n)_m \notin J_m\} = X \setminus X_n \in I.$$

Hence $A \setminus A_n \in \sum_I J_n$. □

The following corollary generalizes Milovich's theorem 0.4 taking $I = \text{fin}$:

Corollary 3.15. *Suppose that I is a deterministic ideal and $U, U_0, U_1, \ldots$ all the the I-p.i.p. Let $W = \sum_U U_n$, then W has the $I \cdot I$-p.i.p and in particular $W \cdot W \equiv_T W \times I^\omega$.*

Proof. All the ultrafilters U, U_0, U_1's satisfy the I-p.i.p and therefore by Proposition 3.14, $\sum_U U_n$ satisfies the $\sum_I I$-p.i.p which is $\text{fin} \cdot \text{fin}$-p.i.p. By theorem 1.2, $I \cdot I \equiv_T I^\omega$. Note that $I \cdot I \subseteq (\sum_U U_n)^*$, and by Theorem 3.6 $I \cdot I$ also deterministic. Hence

$$I^\omega \equiv_T I \cdot I \leq_T \sum_U U_n.$$

Finally, by Proposition 1.6 it follows that $(\sum_U U_n) \cdot (\sum_U U_n) \equiv_T (\sum_U U_n) \times I^\omega$ □

4. QUESTIONS

We collect here some problems which relate to the work of this paper.

Question 4.1. What is the characterization of the I-p.i.p property in terms of Skies and Constellations of ultrapowers from [26]?

Question 4.2. Is the equivalence of Proposition 1.6 true for every ideal I?

Question 4.3. Is almost rapidness an invariant of the Rudin-Blass order? namely $f : \omega \to \omega$ is finite to one and U is almost rapid, must $f_*(U)$ be also almost rapid?

Question 4.4. Is it true that for every $\alpha < \beta < \omega_1$, the class of α-almost-rapid ultrafilters is consistently strictly included in the class of β-almost-rapid ultrafilters?

We conjecture a positive answer to this question and that similar methods to the one presented in Theorem 2.7 under CH should work.

Question 4.5. Does $\mathfrak{d} = \mathfrak{c}$ imply that there is a p-point which is almost-rapid but not rapid?

Question 4.6. Is it consistent that there are no almost-rapid ultrafilters?

Following Miller, a natural model would be adding $\aleph_2$-many Laver reals.

Question 4.7. I there a non-Canjar p-point ultrafilter which does not have almost rapid RK-predecessors?

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