Finals Example- Set Theory fall 2023

MATH 361 (Instructor: Tom Benhamou) Dec 20

Instructions

The midterm duration is 3 hours, and consists of 5 problems, each worth 21 points (The maximal grade is 100). The answers to the problems should be written in the designated areas.

Problems

Problem 1. Let A be any set. Let us define recursively $A_0 = A$ and $A_{n+1} = P(A_n)$. Define $A_\omega = \bigcup_{n<\omega} A_n$. Prove that for every set A and any $n < \omega$, $A_n \prec A_\omega$.

Solution: By inclusion, for every $n < \omega$, $A_n \preceq A_\omega$. Suppose toward a contradiction that there is n such that $A_n \approx A_\omega$.Then $A_n \preceq A_{n+1} \preceq A_\omega \approx A_n$ and therefore by CSB Theorem $A_{n+1} \approx A_n$. However, $A_{n+1} = P(A_n)$, this is a contradiction to Cantor's theorem that for every set A, $A \prec P(A)$.

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Problem 2. (a) Define what a "well ordered set" and "isomorphic well ordered sets" are.

(b) Prove that if $\langle A, <_A \rangle, \langle B, <_B \rangle$ are isomorphic well-ordered sets, then then there is a unique isomorphism $g : A \to B$.

Solution: Suppose that g_1, g_2 are isomorphism. And towards a contradiction suppose that there is x such that $g_1(x) \neq g_2(x)$. Then $D = \{a \in A \mid g_1(x) \neq g_2(x)\} \neq \emptyset$ and therefore there exists $x^* = \min_{<A}(D)$. WLOG assume that $g_1(x^*) <_B g_2(x^*)$, since g_2 is an isomorphism, there is $y \in A$ such that $g_2(y) = g_1(x^*)$ and since g_2 is order-preserving, $y <_A x^*$. We conclude that $g_1(y) < g_1(x^*) = g_2(y)$, hence $g_1(y) \neq g_2(y)$ which implies that $y \in D$, contradicting the minimality of x^*.

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Problem 3. Problem from HW6-HW10 (not from "additional problems")

Solution

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Problem 4. (a) Let $f : \mathbb{N} \to \mathbb{N}$ be any function. we denote by $f^k = f \circ f \ldots \circ f$ the composition of f with itself k-many times and $f^0 = id_{\mathbb{N}}$. Define the relation E_f on $\mathbb{N}$ as follows: $mE_f n$ if and only of $\exists k, f^k(m) = f^k(n)$. Prove that E_f is an equivalence relation.

(b) Consider the equivalence relation E on ${}^{\mathbb{N}}\mathbb{N}$ defined by fEg iff $f \restriction \mathbb{N}_{even} = g \restriction \mathbb{N}_{even}$ (no need to prove it). Compute the cardinality of $[id_{\mathbb{N}}]_E$

Solution

By definition we have that $[id_{\mathbb{N}}]_E = \{f \in {}^{\mathbb{N}}\mathbb{N} \mid \forall n \in \mathbb{N}_{even},\ f(n) = n\}$. Hence it is possible to find a bijection of $[id_{\mathbb{N}}]_E$ with ${}^{\mathbb{N}_{odd}}\mathbb{N}$ defined by $F : {}^{\mathbb{N}_{odd}}\mathbb{N} \longrightarrow [id_{\mathbb{N}}]_E$

$$F(f)(n) = \begin{cases} n & n \in \mathbb{N}_{even} \\ f(n) & n \in \mathbb{N}_{odd} \end{cases}$$

Hence $|[id_{\mathbb{N}}]_E| = |{}^{\mathbb{N}_{odd}}\mathbb{N}| = \aleph_0^{\alpha_0} = 2^{\aleph_0}$.

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Problem 5. (a) Given the integers $\mathbb{Z}$ together with the arithmetic operations $+, -, \cdot$, present the definition of $\mathbb{Q}, +$.

(b) Prove that rational addition does not depend on the choice of representatives and that it is commutative. Namely, $q + p = p + q$. You can assume the usual properties of addition and multiplication of integers.

Solution Recall that addition is defined by

$$[\langle z, z' \rangle] + [\langle t, t' \rangle] = [\langle zt + z't', z't' \rangle]$$

The first part was proven in class. To see the commutativity, we use the commutativity of + and $\cdot$ for integers:

$$[\langle z, z' \rangle] + [\langle t, t' \rangle] = [\langle zt + z't', z't' \rangle] = [\langle tz + t'z', t'z' \rangle] = [\langle t, t' \rangle] + [\langle z, z' \rangle]$$