Midterm example

Problem 1 For each of the following determine true/false. If false provide a counterexample.

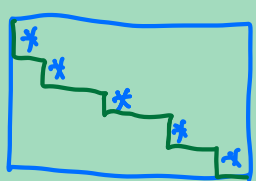
(a) 3 vectors in $\mathbb{R}^5$ are linearly independent. True \ False

Counterexample: $\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$ or $\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix}$

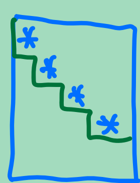
(b) 7 vectors in $\mathbb{R}^6$ are linearly dependent. True \ False

More than n vectors in $\mathbb{R}^n$ cannot be linearly independent, because

Columns independent $\Leftrightarrow$ the Echelon form has a leading entry in each column



not LI



LI

(c) The columns of a 5×5 matrix with two equal rows are linearly depent. True \ False

The echelon form will have a row of zeros, so there cannot be a leading entry in each column.

Problem 2 Determine whether $\begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}, \begin{bmatrix}3\\0\\3\end{bmatrix}$ are linearly independent and/or spanning.

Solution: Calculate the echelon form. LI $\Leftrightarrow$ leading entry in each column

Spanning $\Leftrightarrow$ leading entry in each row

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ -1 & -1 & 1 & 0 \\ 1 & 0 & -1 & 3 \end{bmatrix} \xrightarrow[③=③+②]{②=②+①} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & 0 & 3 \end{bmatrix} \xrightarrow{③=③+②} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

LI: No   Spanning: Yes

Problem 3 Consider the system $\begin{cases} x+y+z=1 \\ x+y+\frac{\alpha}{2}z=2 \\ x+y+\alpha z=1 \end{cases}$ where $\alpha \in \mathbb{R}$ is a parameter.

Determine for which α it has no solution / unique solution / ∞-many solutions.

Write the general solution in parametric representation for any α such that there is a solution.

Solution: Let's try to calculate the echelon form of augmented matrix.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & \frac{\alpha}{2} & 2 \\ 1 & 1 & \alpha & 1 \end{bmatrix} \xrightarrow[③=③-①]{②=②-①} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & \frac{\alpha}{2}-1 & 1 \\ 0 & 0 & \alpha-1 & 0 \end{bmatrix}$$

Case 1 $\alpha=1$

Then we have $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -\frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x+y+z=1 \\ -\frac{1}{2}z=1 \\ 0=0 \end{cases} \Rightarrow \begin{cases} x=3-y \\ y \text{ is free} \\ z=-2 \end{cases}$ ∞-many solutions

Case 2 $\alpha \neq 1$

Then from $(\alpha-1)z=0$ we get $z=0$. But then $\left(\frac{\alpha}{2}-1\right)z$ cannot be 1.

So there is no solution.

Linear transformations

Goal: linear transformations $\Longleftrightarrow$ matrices

f is a transformation/function/mapping from A to B if:

1. it assigns to every $a \in A$ an element $b \in B$
2. the element $b \in B$ assigned to $a \in A$ is unique, and is denoted by $f(a)$

   (think about a deterministic algorithm)

We write $f: A \to B$ to indicate f is a function from A to B. We call A the domain of f, denoted $\mathrm{dom}(f)$, and B the co-domain of f, denoted $\mathrm{codom}(f)$.

The collection of all b that are assigned to some a, namely $\{b \in B : f(a) = b \text{ for some } a \in A\}$ is called the range of f, denoted $\mathrm{ran}(f)$.

Examples of functions:

(1) $f(x) = x^2$, $f(x) = e^x$, $f(x) = \begin{cases} x+1, & x \le 0 \\ x-1, & x > 0 \end{cases}$ are functions from $\mathbb{R}$ to $\mathbb{R}$

(2) $f(x,y) = x+y$, $f(x,y) = \ln(x^2+y^2+1)$ are functions from $\mathbb{R}^2$ to $\mathbb{R}$

$f(x,y,z) = x^2 + 2y + 5z^3$, $f(x,y,z) = 0$ are functions from $\mathbb{R}^3$ to $\mathbb{R}$

(3) $f(x) = (x, x^2, x^3)$ is a function from $\mathbb{R}$ to $\mathbb{R}^3$, also called a parametrized curve

$f(x,y) = (x+y, x-y, x^2+y^2)$ is a function from $\mathbb{R}^2$ to $\mathbb{R}^3$, also called a parametrized surface

(4) A function from $\mathbb{R}^n$ to $\mathbb{R}^m$ is basically m functions from $\mathbb{R}^n$ to $\mathbb{R}$

(5) Given an $m \times n$ matrix A, we define a matrix transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ by $T_A(\vec{x}) = A\vec{x}$. For example if $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix}$, then

$$T_A(x,y,z) = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+2y \\ x+y-z \end{bmatrix} = (x+2y, x+y-z)$$

$$T_A(1,0,1) = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (1,0)$$

(6) Consider the identity matrix $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. T_{I_3} is the identity map on $\mathbb{R}^3$, i.e., it maps every vector $\vec{x} \in \mathbb{R}^3$ to itself.

Example 1 p.68 Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$. Denote T_A by T, so

$$T(\vec{x}) = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

(a) Find $T(\vec{u})$ (b) Find an $\vec{x} \in \mathbb{R}^2$ whose image under T is $\vec{b}$

(c) Is there more than one $\vec{x}$ whose image under T is $\vec{b}$?

(d) Determine if $\vec{c}$ is in $\operatorname{ran}(T)$

Solution: (a) $T(\vec{u}) = \begin{bmatrix} 2 - 3\cdot(-1) \\ 3\cdot 2 + 5\cdot(-1) \\ -2 + 7\cdot(-1) \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$

(b) We need to solve $\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \xrightarrow[③ = ③ + ①]{② = ② - 3①} \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix} \xrightarrow[③ = \frac{1}{2}③]{② = \frac{1}{7}②} \begin{bmatrix} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix} \xrightarrow{③ = ③ - ②} \begin{bmatrix} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{cases} x_1 - 3x_2 = 3 \\ 2x_2 = -1 \end{cases} \longrightarrow \begin{cases} x_1 = \frac{3}{2} \\ x_2 = -\frac{1}{2} \end{cases}$$ So $\begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}$ is as desired

(c) From (b) we see $\vec{x} = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}$ is the only solution to $T(\vec{x}) = \vec{b}$

(d) Basically same as (b) (except in (b) we are told there is a solution)

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \xrightarrow[③ = ③ + ①]{② = ② - 3①} \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \xrightarrow[③ = \frac{1}{4}③]{② = \frac{1}{7}②} \begin{bmatrix} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{③ = ③ - \frac{1}{2}②} \begin{bmatrix} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{5}{2} \end{bmatrix}$$

The last row says $0 \cdot x_1 + 0 \cdot x_2 = \frac{5}{2}$, which is impossible. So there is no solution to $T(\vec{x}) = \vec{c}$, i.e., $\vec{c} \notin \operatorname{ran}(T)$.

Definition A transformation $f: \mathbb{R}^n \to \mathbb{R}^m$ is *linear* if for any $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$:

1. $f(\vec{u} + \vec{v}) = f(\vec{u}) + f(\vec{v})$
2. $f(\alpha \vec{u}) = \alpha f(\vec{u})$

Linear functions are easy to study. That's why when we need to deal with non-linear functions, we often consider their linearizations.

Matrix transformation is linear (on Wednesday) and the converse is also true.

Let's verify this for 2×2 matrix.

Rule 1:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix} = \begin{bmatrix} a(u_1+v_1)+b(u_2+v_2) \\ c(u_1+v_1)+d(u_2+v_2) \end{bmatrix}$$

$$= \begin{bmatrix} au_1 + av_1 + bu_2 + bv_2 \\ cu_1 + cv_1 + du_2 + dv_2 \end{bmatrix} = \begin{bmatrix} au_1+bu_2 \\ cu_1+du_2 \end{bmatrix} + \begin{bmatrix} av_1+bv_2 \\ cv_1+dv_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Rule 2:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(r\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} ru_1 \\ ru_2 \end{bmatrix} = \begin{bmatrix} aru_1+bru_2 \\ cru_1+dru_2 \end{bmatrix} = \begin{bmatrix} r(au_1+bu_2) \\ r(cu_1+du_2) \end{bmatrix} = r\begin{bmatrix} au_1+bu_2 \\ cu_1+du_2 \end{bmatrix} = r\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right)$$

Example 4 p.71 Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\vec{x}) = r\vec{x}$ where $r \in \mathbb{R}$ is some fixed number. Then T is a linear transformation, because it is a matrix transformation: $T(\vec{x}) = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\vec{x}$. This is called dilation by r.

Also see Examples 2 (projection) 3 (shear transformation) and 5 (rotation).

Useful fact (a) If $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear, then $T(\alpha\vec{u} + \beta\vec{v}) = \alpha T(\vec{u}) + \beta T(\vec{v})$ for any $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$

(b) More generally, $T(c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_p\vec{v_p}) = c_1T(\vec{v_1}) + c_2T(\vec{v_2}) + \cdots + c_pT(\vec{v_p})$

Solution: (a) $T(\alpha\vec{u} + \beta\vec{v}) = T(\alpha\vec{u}) + T(\beta\vec{v})$ apply rule 1

$= \alpha T(\vec{u}) + \beta T(\vec{v})$ apply rule 2 twice

(b) Let's do it for $p = 4$

$T(\underline{c_1\vec{v_1}} + \underline{c_2\vec{v_2} + c_3\vec{v_3} + c_4\vec{v_4}})$ apply rule 1 to the two underlined vectors

$= T(c_1\vec{v_1}) + T(\underline{c_2\vec{v_2}} + \underline{c_3\vec{v_3} + c_4\vec{v_4}})$

$= T(c_1\vec{v_1}) + T(c_2\vec{v_2}) + T(\underline{c_3\vec{v_3}} + \underline{c_4\vec{v_4}})$

$= T(c_1\vec{v_1}) + T(c_2\vec{v_2}) + T(c_3\vec{v_3}) + T(c_4\vec{v_4})$

$= c_1T(\vec{v_1}) + c_2T(\vec{v_2}) + c_3T(\vec{v_3}) + c_4T(\vec{v_4})$ apply rule 2 four times