COMMUTATIVITY OF COFINAL TYPES

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ABSTRACT. We study two problems regrading the cofinal type of Fubini sums of ultrafilters and the commutativity of cofinal types with respect to Fubini products of ultrafilters. Our main result is that the cofinal type of sums of ultrafilters is the infimum Tukey-type of a natural class of cofinal types. We then use this result to prove that the class of ultrafilters U such that $U \cdot U \equiv_T U \times \omega^\omega$ is closed under Fubini sums. We conclude that for a large class of ultrafilters, which includes all known examples, commutativity of cofinal types holds.

0. INTRODUCTION

In this paper, we study two questions regarding the cofinal type of ultrafilters. By the cofinal type of an ultrafilter U, we mean the Tukey-equivalence class of the ordered set $(U, \supseteq)$. Before we reveal those question, let us lay out the definition of Tukey equivalence and the Tukey order [18], which traces back to the study of Moore-Smith convergence of nets in Topology. Given two posets, $(P, \leq_P)$ and $(Q, \leq_Q)$ we say that $(P, \leq_P) \leq_T (Q, \leq_Q)$ if there is a *cofinal map* $f : Q \to P$, that is, $f(\mathcal{B})$ is cofinal in P whenever $\mathcal{B} \subseteq Q$ is cofinal. Dually, Schmidt [17] showed that the existence of a cofinal map is equivalent to the existence of a map $f : P \to Q$, which sends an unbounded set $\mathcal{A}$ in P to an unbounded set $f(\mathcal{A})$ in Q. These are called *unbounded maps* or *Tukey reductions*. We say that P and Q are *Tukey equivalent*, and write $P \equiv_T Q$, if $P \leq_T Q$ and $Q \leq_T P$; the equivalence class $[P]_T$ is called the *Tukey type* or *cofinal type* of P.

The problems we are interested in regard the correspondence between the Tukey order and Fubini sums and products: Given ultrafilters $U, V_0, V_1, V_2, ...$ on ω, the *Fubini sum* over U, is an ultrafilter on $\omega \times \omega$, denote by $\sum_U V_n$ consisting of all $A \subseteq \omega \times \omega$ such that for U-many n's, then n^{th} fiber of $\{m \mid (n,m) \in A\}$ is in V_n. The *Fubini product* of U and V, denoted by $U \cdot V$ is defined as the Fubini sum over U of the constant sequence $V_n = V$.

Here are the two main question this paper is concerned with:

Main Question 1. What is the Tukey-type of $\sum_U V_n$ in terms of the Tukey-types of $U, V_0, V_1, ...$?

Main Question 2. If the Tukey order on ultrafilter commutative? Namely, is it ture that for any two ultrafilters U, V, we have $U \cdot V \equiv_T V \cdot U$?

There is a good reason to believe that Main Question 1 is achievable, indeed the Tukey-type of Fubini products of filters factores nicely [15, 9]:

Theorem 0.1 (Milovich, Dobrinen-Todorcevic)**.** *Let F, G are κ-filters*[1]*, then $F \cdot G \equiv_T F \times G^\kappa$ and in particular $F \cdot F \equiv_T F^\kappa$.*

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[1] i.e. κ-complete filters over κ.

In the previous theorem and throughout this paper, F^κ is a (full support) κ-product of copies of F, with the order defined pointwise. Also, Dobrinen and Todorcevic showed that $\sum_U V_n \leq_T U \times \prod_{n<\omega} V_n$ (see Theorem 2.1). Hence it is tempting to conjecture that $\sum_U V_n \equiv_T U \times \prod_{n<\omega} V_n$. It turns out that in the must general case, this formula is not true going to be true (see Example 2.3). Nonetheless, our first theorem shows that under some assumption on the ultrafilters V_n, it is:

Theorem. *Suppose that $U, V_0, V_1, ...$ are an ultrafilter over ω. Suppose that there is a set $X_0 \in U$, such that for every $n \leq m \in X_0$, $V_n \leq_T V_m$. Then*

$$U \times \prod_{n\in X_0} V_n \equiv_T \sum_U V_n$$

Note that this theorem captures Theorem 0.1 as well. To deal with the general case, we first observe (see Fact 2.2) that better approximations of the cofinal type of $\sum_U V_n$ are given by any cofinal type in the set

$$\mathcal{B}(U, \langle V_n \mid n < \omega\rangle) := \{U \times \prod_{n\in X} V_n \mid X \in U\}.$$

This simple observation motivates our principal conjecture, which addresses Main Question 1:

Conjecture 0.2. Let $U, V_0, V_1, ...V_n$ be ultrafilters on ω, the Tukey type of $\sum_U V_n$ is the greatest lower bound in the Tukey order of the set $\mathcal{B}(U, \langle V_n \mid n < \omega\rangle)$.

By greatest lower bound we mean that if $\mathbb{P}$ is any directed order such that for all $X \in U$, $\mathbb{P} \leq_T U \times \prod_{n\in X} V_n$, then $\mathbb{P} \leq_T \sum_U V_n$. For example, this is true in the set up of the previous theorem:

Theorem. *Suppose that $X_0 \in U$, and $\sum_U V_\alpha \equiv_T U \times \prod_{\alpha\in X_0} V_\alpha$. Then $\sum_U V_\alpha$ is the greatest lower bound of $\mathcal{B}(U, \langle V_n \mid n < \omega\rangle)$.*

We then provide an example where the assumption of the previous theorem fails:

Theorem. *There consistently exists $U, V_0, V_1, ...$ on ω such that for every $X \in U$,*

$$U <_T \sum_U V_n <_T U \times \prod_{n\in X} V_n,$$

yet $\sum_U V_n$ is the greatest lower bound of $\mathcal{B}(U, \langle V_n \mid n < \omega\rangle)$.

A key idea here is that $\sum_U V_n$ a lower bound of $\mathcal{B}(U, \langle V_n \mid n < \omega\rangle)$ in a uniform way in the sense that there is a system of cofinal maps which coheres (see Definition 2.6). We show that given a coherent system of monotone cofinal maps from $\mathcal{B}(U, \langle V_n \mid n < \omega\rangle)$ to an order $\mathbb{P}$, can be amalgamated to a single monotone cofinal map from the $\sum_U V_n$ to $\mathbb{P}$. Thus, our main result shows that $\sum_U V_n$ is indeed a greatest lower bound among the posets which uniformly below $\mathcal{B}(U, \langle V_n \mid n < \omega\rangle)$:

Theorem. *Suppose that $\mathbb{P}$ is a complete order[2]. Then $\mathbb{P}$ is uniformly below $\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda\rangle)$ if and only if $\sum_U V_\alpha \geq_T \mathbb{P}$.*

We then apply this theorem to show some progress on the Main Question 2, suggested in [3]. This problem was studied in the past. Below is a list of the known results:

- If U, V are p-points, then $U \cdot V \equiv_T V \cdot U$ (Milovich [15]).

[2]Note that any order can be cofinally embedded (and therefore Tukey equivalent) into a complete order, i.e. the Boolean algebra of regular open cuts (see [11, Thm. 7.13]).

- If U, V are κ-complete for $\kappa > \omega$ then $U \cdot V \equiv_T V \cdot U$ (Benhamou-Dobrinen [2]).
- For all U, V, $(U \cdot U) \cdot (V \cdot V) \equiv_T (V \cdot V) \cdot (U \cdot U)$ (Benhamou-Dobrinen [3])

The commutativity of cofinal types stands in sharp contrast to the Rudin-Keisler ordering which is known to be highly non commutative with respect to Fubini product[3]. On measurable cardinals, the situation is even more dramatic, due to a theorem of Solovey (see [12, Thm. 5.7]) if U, W are κ-complete ultrafilters on κ the $U \cdot W \equiv_T W \cdot U$ if and only if $W \equiv_{RK} U^n$ for some n or vise versa. Recently, Goldberg [10] examined situations of commutativity with respect to several product operations on countably complete ultrafilters.

Our contribution is to study classes of the form

$$\mathcal{E}_D := \{U \mid U \cdot U \equiv_T U \times D\}$$

We will show that any two ultrafilter in such a class Tukey-commute. Then our main result is to show that following:

Theorem. *$\mathcal{E}_{\omega^\omega}$ is closed under Fubini sums.*

The class $\mathcal{E}_{\omega^\omega}$ is extremely large. It includes:

(I) Tukey-top ultrafilters.
(II) p-points ([15, Thm. 5.4]) and their sums (Theorem above).
(III) Every ultrafilter satisfying $U \cdot U \equiv_T U$ (Claim 3.4), hence:
  (a) Ultrafilters of the form $U \cdot U$ (Theorem 0.1).
  (b) Stable ordered union ultrafilters [3, Thm. 4.2].
  (c) Generic ultrafilters for $P(\omega)/I$ where $I^\omega \equiv_T I$, and $\Vdash_{P(\omega)/I}$ "$I \leq_T \dot{G}$"[4] [1, Cor. 1.19]
  (d) Ultrafilters arising from topological Ramsey spaces [5, Thm 4.6]

To see why the above ultrafilters cover all knwon example, note that an ultrafilter which does not fall under (I),(II),(III) above must be either:

(1) Basically generated and not an iterated sum of p-points (see Question 26 in [9]). Or
(2) non-Tukey-top which is not basically generated– the known examples for such ultrafilters are generic ultrafilters for $P(\omega)/\mathrm{fin}^{\otimes\alpha}$ [4, 7].

Note that by the recent result of Cancino and Zapletal [6], it is consistent that every ultrafilter on ω is Tukey-top hence therefore the commutativity of cofinal types is indeed consistent.

1. Preliminaries

In this section we set up notations, and provide basic facts and definitions regarding ultrafilters and the Tukey order. Given a set $X \subseteq \omega$, such that $|X| = \alpha \leq \omega$, we denote by $\langle X(\beta) \mid \beta < \alpha \rangle$ be the increasing enumeration of X. The principal operation we are considering in this paper is the Fubini/tensor sums and products of ultrafilters.

Definition 1.1. Suppose that F is a filter over an infinite set X and for each $x \in X$, G_x is a filter over an infinite set Y_x. We denote by $\sum_F G_x$ the filter over $\bigcup_{x \in X}\{x\} \times Y_x$, defined by

$$A \in \sum_F G_x \text{ if and only if } \{x \in X \mid (A)_x \in G_x\} \in F$$

[3]For example if U, W are non-isomorphic Ramsey ultrafilters then $U \cdot W \not\equiv_{RK} W \cdot U$. Just otherwise, by a theorem of Rudin (see for example [12, Thm. 5.5], U, W should be Rudin-Frolík (and therefore Rudin-Keisler) comparable, contradicting the RK-minimality of Ramsey ultrafilters.

[4]for example $\mathrm{fin}^{\otimes\alpha}$.

where $(A)_x = \{y \in Y_x \mid \langle x, y\rangle \in A\}$ is the x^{th}-*fiber* of A. If for every x, $G_x = G$ for some fixed V over a set Y, then $F \cdot G$ is defined as $\sum_F G$, which is a filter over $X \times Y$. F^2 denotes the filter $F \cdot F$ over $X \times X$.

We distinguish here between $F \cdot G$ and $F \times G$ which is the cartesian product of F and G with the pointwise order [5].

For any set X and cardinal λ $[X]^\lambda$ denotes the set of all subsets of X of cardinality λ, and $[X]^{<\lambda} = \bigcup_{\alpha<\lambda}[X]^\alpha$. In particular we set $\mathrm{fin} = [\omega]^{<\omega}$, and $\mathrm{FIN} = \mathrm{fin} \setminus \{\emptyset\}$.

Definition 1.2. Let F be a filter over a regular cardinal $\kappa \geq \omega$.

(1) F is *λ-complete* if F is closed under intersections of less than λ many of its members.
(2) F is *selective* if for every function $f : \kappa \to \kappa$, there is an $X \in F$ such that $f \restriction X$ is either constant or one-to-one.
(3) F is *rapid* if for each normal function $f : \kappa \to \kappa$ (i.e. increasing and continuous), there exists an $X \in F$ such that $\mathrm{otp}(X \cap f(\alpha)) \leq \alpha$ for each $\alpha < \kappa$. (i.e. bounded pre-images), there is an $X \in F$ such that $|f^{-1}(\{\alpha\}) \cap X| \leq \alpha$ for every $\alpha < \kappa$.
(4) F is a *p-point* if whenever $f : \kappa \to \kappa$ is unbounded[6] on a set in F, it is almost one-to-one mod F, i.e. there is an $X \in F$ such that for every $\gamma < \kappa$, $|f^{-1}[\gamma] \cap X| < \kappa$.

a *κ-filter* is a uniform[7], κ-complete filter.

The following fact is well known.

Fact 1.3. Suppose that U, V_α are ultrafilters on $\kappa \geq \omega$ where each V_α is uniform. Then $\sum_U V_\alpha$ is not a p-point.

Indeed the function π_1 – the projection to the first coordinate – is never almost one-to-one on a set in $X \in \sum_U V_\alpha$. Given a function $f : A \to B$, for $X \subseteq A$ we let $f(X) = \{f(x) \mid x \in X\}$, for $Y \subseteq B$ we let $f^{-1}Y = \{x \in X \mid f(x) \in Y\}$, and let $\mathrm{rng}(f) = f(A)$.

Definition 1.4. Let F, G be filters on X, Y resp. We say that F is *Rudin-Keisler* below G, denoted by $F \leq_{RK} G$, if there is a *Rudin-Keisler* projection $f : Y \to X$ such that

$$f_*(G) := \{A \subseteq X \mid f^{-1}(A) \in G\} = F$$

We say that are RK-isomorphic, and denote it by $F \equiv_{RK} G$ if there is a bijection f such that $f_*(F) = G$.

It is well known that if $F \leq_{RK} G \wedge G \leq_{RK} F$ then $F \equiv_{RK} G$ and that $F, G \leq_{RK} F \cdot G$ via the projection to the first and second coordinates respectively. Also, the Rudin-Keisler order implies the Tukey order. A selective ultrafilter over κ is characterized as being Rudin-Keisler minimal among κ-ultrafilters.

Next, let us record some basic terminology and facts regarding cofinal types. Given two directed partially ordered sets $\mathbb{P}, \mathbb{Q}$, the Cartesian product $\mathbb{P} \times \mathbb{Q}$ ordered pointwise, is the least upper bound of $\mathbb{P}, \mathbb{Q}$ in the Tukey order (see [8]). It follows that $F \times G \leq_T F \cdot G$. More generally, for partially ordered sets $\mathbb{P}_i = (P_i, \leq_i)$ for $i \in I$, we denote by $\prod_{i\in I}(\mathbb{P}_i, \leq_i) = (\prod P_i, \leq)$, where $\prod_{i\in I} P_i = \{f \mid \mathrm{dom}(f) = I \text{ and } \forall i,\ f(i) \in P_i\}$ is equipped with the everywhere domination order, namely, $f \leq g$ iff for all $i \in I$, $f(i) \leq_i g(i)$. If the order is clear from the context we omit it and just write $\prod_{i\in I} \mathbb{P}_i$. This is the case when $\mathbb{P}_i = U_i$ is

[5] There are papers which consider the filter $\{A \times B \mid A \in F, B \in G\}$ and denote it by $F \times G$, this filter will not be considered in this paper so there is no risk of confusion.

[6] Namely, $f^{-1}[\alpha] \notin F$ for every $\alpha < \kappa$

[7] i.e for every $X \in U$, $|X| = \kappa$.

a filter ordered by reversed inclusion of an ideal ordered by inclusion. If for every $i \in I$, $\mathbb{P}_i = \mathbb{P}$ we simply write $\mathbb{P}^I$.

2. On the Cofinal Types of Fubini Sums of Ultrafilters

The purpose of this section is to study the cofinal type of sums of ultrafilter. The following theorem [9] provides the starting point:

Theorem 2.1 (Dobrinen-Todorcevic)**.** *Let F be a filter over λ and $(G_\alpha)_{\alpha<\lambda}$ be any sequence of filters. Then:*

$$\sum_F G_\alpha \leq_T F \times \prod_{\alpha<\lambda} G_\alpha$$

More generally, we have:

Fact 2.2. Let U be an ultrafilter over $\lambda \geq \omega$ and U_α on δ_α. For every $X \in U$, we have $\sum_U V_\alpha \leq_T U \times \prod_{\alpha\in X} V_\alpha$.

Proof. Since the set $\mathcal{X} \subseteq \sum_U V_\alpha$, of all Y such that $\pi_1(Y) \subseteq X$ is a cofinal in $\sum_U V_\alpha$a, the map $F : U \times \prod_{\alpha\in X} V_\alpha \to \sum_U V_\alpha$ defined by

$$F(\langle Z, \langle A_\alpha \mid \alpha \in X\rangle\rangle) = \bigcup_{\alpha\in Z\cap X} \{\alpha\} \times A_\alpha$$

is monotone and cofinal. □

Example 2.3. Suppose that U and V are Tukey incomparable ultrafilters on ω, and $U \equiv_T U \cdot U$. This situation is obtained for example under $Cov(\mathcal{M}) = \mathfrak{c}$ [8]. The incomparability ensures that $U \times V >_T U$. Let $V_0 = V$ and $V_n = U$ for $n > 0$. Then

$$\sum_U V_n = U \cdot U \equiv_T U <_T U \times V \leq_T U \times \prod_{n<\omega} V_n. \tag{1}$$

Assume that U concentrates on $\mathbb{N}_{even}$, let

$$V'_n = \begin{cases} U & n = 2k \\ V & n = 2k+1 \end{cases} \text{ and } V''_n = \begin{cases} U & n = 2k \\ V & n = 2k+1 \end{cases}.$$

Then

$$\sum_U V'_n = U \cdot U <_T U \times V = \sum_U V''_n \tag{2}$$

Example (1) illustrates the fact that the sum is insensitive to a neglectable set of coordinates, while the product is. Example (2) illustrates that the product is insensitive to permutations of the indexing set, while the sum is.

First, let us present a theorem which deals with a specific situation of sums, but is general enough to captures Theorem 0.1 as a special case [9]:

Theorem 2.4. *Suppose that U, V_n are an ultrafilter over ω. Suppose that there is a set $X_0 \in U$, such that for every $n \leq m \in X_0$, $V_n \leq_T V_m$. Then*

$$U \times \prod_{n\in X_0} V_n \equiv_T \sum_U V_n$$

[8]By Ketonen [13], this assumption implies that there are $(2^c)^+$-many distinct selective ultrafilters. Then there are two Tukey incomparable selective ultrafilters and by Dobrinen and Todorcevic [9], $U \cdot U \equiv_T U$ for any selective ultrafilter.

[9]Taking $V_n = V$ for all n.

Proof. For the easy direction, use Fact 2.2. Also $U \leq_T \sum_U V_n$ and by the Tukey-minimality of the cartesian product, it remains to show $\prod_{n\in X_0} V_n \leq_T \sum_U V_n$. First define for $n \in X_0$, $n^+ = \min(X_0 \setminus n+1)$ and let $f_{n^+,n} : V_{n^+} \to V_n$ monotone and cofinal. Denote by $n^{+k} = (n^{+(k-1)})^+$ the k^{th} successor of n in X_0 and let

$$f_{n^{+k},n} = f_{n^{+k},n^{+(k-1)}} \circ \ldots \circ f_{n^{+2},n^+} \circ f_{n^+,n}.$$

Moreover, let $f_{n,n} = \mathrm{id}_{V_n}$. Hence each $f_{m,n} : V_m \to V_n$ is monotone cofinal, and if $k \in X_0 \cap [n,m]$ then $f_{m,n} = f_{k,n} \circ f_{m,k}$.

Define $\mathcal{X} \subseteq \sum_U V_n$ to consist of all $A \in \sum_U V_n$ in standard form[10] such that $\pi_0(A) \subseteq X_0$, and for all $n < m$ in $\pi_0(A)$, $f_{m,n}((A)_m) \subseteq (A)_n$.

Define $F : \mathcal{X} \to \prod_{n\in X_0} V_n$ by setting

$$F(A)_k = f_{m^A_k,k}((A)_m), \text{ where } m^A_k = \min(\pi(A) \setminus k) \geq k.$$

The following claim concludes the proof:

Claim 2.5.
(1) *$\mathcal{X}$ is a base for $\sum_U V_n$.*
(2) *F is monotone and cofinal.*

Proof of Claim. To see (1), let $B \in \sum_U V_n$. Find $A' \subseteq B$ in standard form, and let $X = \pi(A')$. Define a sequence A_n be induction on $n \in X$. Set $A_{\min(X)} = (A')_{\min(X)}$. Suppose that $m \in X$ and $A_k \in V_k$ is defined for all $k \in X \cap m$. For each $k \in X \cap m$, find $C_{m,k} \in V_m$ such that $f_{m,k}(C_{m,k}) \subseteq A_k$. Define $A_m = (A')_m \cap (\bigcap_{k\in X\cap m} C_{m,k})$. By monotonicity, $f_{m,k}(A_m) \subseteq A_k$ and $A = \bigcup_{k\in X}\{k\} \times A_k$ is as wanted.

To see (2), if $A \subseteq B$, then for every $k \in X_0$, $m^A_k > m^B_k$, hence

$$\begin{aligned} F(A)_k &= f_{m^A_k,k}((A)_{m^A_k}) \\ &\subseteq f_{m^k_A,k}((B)_{m^A_k}) \\ &= f_{m^B_k,k}(f_{m^A_k,m^B_k}((B)_{m^A_k})) \\ &\subseteq f_{m^B_k,k}((B)_{m^B_k}) = F(B)_k. \end{aligned}$$

To see it is cofinal, take any $\langle A_n \mid n \in X_0\rangle \in \prod_{n\in X_0} V_n$. By the construction of (1), we can find $A^*_n \subseteq A_n$, $A^*_n \in V_n$ such that if $n < m$ are in X_0, then $f_{m,n}(A^*_m) \subseteq A^*_n$. Let $A^* = \bigcup_{n\in X_0}\{n\} \times A^*_n$. Then $A^* \in \mathcal{X}$, and $F(A^*) = \langle A^*_n \mid n \in X_0\rangle$. □

□

Parts of our theory applies to ultrafilter over arbitrary cardinal. Thus our initial assumption is that U is a λ-ultrafilter for $\lambda \geq \omega$ and $\langle V_\alpha \mid \alpha < \lambda\rangle$ is a sequence of ultrafilters such that each V_α is a δ_α-ultrafilter where $\delta_\alpha \geq \omega$. Towards our first result, consider the set

$$\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda\rangle) = \{U \times \prod_{\alpha\in X} V_\alpha \mid X \in U\}$$

ordered by the Tukey order. This is clearly a downward-directed set. Our goal is to prove that in some sense, $\sum_U V_\alpha$ is the greatest lower bound of $\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda\rangle)$.

Consider the maps

$$\pi_X : U \times \prod_{\alpha\in X} V_\alpha \to \sum_U V_\alpha, \quad \pi_{X,Y} : U \times \prod_{\alpha\in X} V_\alpha \to U \times \prod_{\alpha\in Y} V_\alpha$$

[10] A set $B \in \sum_U V_\alpha$ is said to be in *standard form* if for every $\alpha < \lambda$, either $(B)_\alpha = \emptyset$ or $(B)_\alpha \in V_\alpha$.

Defined for $X, Y \in U$ where $Y \subseteq X$ defined by

$$\pi_X(\langle Z, \langle A_\alpha \mid \alpha \in X\rangle\rangle) = \bigcup_{\alpha\in X\cap Z} \{\alpha\}\times A_\alpha \text{ and}$$

$$\pi_{X,Y}(\langle Z, \langle A_\alpha \mid \alpha \in X\rangle\rangle) = \langle Z, \langle A_\alpha \mid \alpha \in Y\rangle\rangle.$$

Then

(1) π_X is monotone cofinal and $\mathrm{rng}(\pi_X)$ is exactly all the sets $B \in \sum_U V_\alpha$ in standard form such that $\pi(B) \subseteq X$.
(2) $\pi_{X,Y}$ is monotone cofinal (infact onto).
(3) $\pi_Y \circ \pi_{X,Y}(C) \subseteq \pi_X(C)$.

Suppose that $\sum_U V_\alpha \geq_T \mathbb{P}$. Recall that if $\mathbb{P}$ is complete[11] (e.g. $(F, \supseteq)$ where F is a filter or any product of complete orders), then $\mathbb{Q} \geq_T \mathbb{P}$ implies that there is a monotone[12] cofinal map $f : \mathbb{Q} \to \mathbb{P}$. Suppose that $\mathbb{P}$ is complete and let $g : \sum_U V_\alpha \to \mathbb{P}$ be monotone cofinal. Define $f_X = g\circ\pi_X$. Then f_X is monotone cofinal from $U \times \prod_{\alpha\in X} V_\alpha$ to $\mathbb{P}$. Moreover, we have that if $Y \subseteq X$ then

$$f_Y(\pi_{X,Y}(C)) = g(\pi_Y(\pi_{X,Y}(C)) \geq_{\mathbb{P}} g(\pi_X(C)) = f_X(C)$$

Definition 2.6. A sequence of monotone cofinal maps

$$\langle f_X : U \times \prod_{\alpha\in X} V_\alpha \to \mathbb{P} \mid X \in U\rangle$$

if said to be *coherent* if

$$(\dagger)\ \text{ whenever } Y \subseteq X, \text{ and } C \in U \times \prod_{\alpha\in X} V_\alpha,\ f_Y(\pi_{X,Y}(C)) \geq_{\mathbb{P}} f_X(C).$$

A poset $\mathbb{P}$ is said to be *uniformly below* $\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda\rangle)$ if there is a coherent sequence of monotone cofinal maps $\langle f_X : U \times \prod_{\alpha\in X} V_\alpha \to \mathbb{P} \mid X \in U\rangle$.

The following theorem says that $\sum_U V_\alpha$ is the greatest lower bound among all the posts uniformly below $\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda\rangle)$.

Theorem 2.7. *Suppose that $\mathbb{P}$ is a complete order. Then $\mathbb{P}$ is uniformly below $\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda\rangle)$ if and only if $\sum_U V_\alpha \geq_T \mathbb{P}$.*

Proof. From right to left was already proven in the paragraph before Definition 2.6. Let us prove from left to right. Let $\langle f_X \mid X \in U\rangle$ be the sequence witnessing that $\mathbb{P}$ is uniformly below $\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda\rangle)$. Let $\mathcal{X} \subseteq \sum_U V_\alpha$ be the usual cofinal set consisting of sets is a standard form. Define $F : \mathcal{X} \to \mathbb{P}$ monotone and cofinal,

$$F(A) = f_{\pi_1(A)}\Big(\Big\langle \pi_1(A), \langle (A)_\alpha \mid \alpha \in \pi(A)\rangle\Big\rangle\Big)$$

To see that F is monotone, let $A, B \in \mathcal{X}$ such that $A \subseteq B$. Define the auxiliary sequence $X_\alpha = \begin{cases} (A)_\alpha & \alpha \in \pi_1(A) \\ (B)_\alpha & \alpha \in \pi_1(B)\setminus\pi_1(A)\end{cases}$. Note that $X_\alpha \subseteq (B)_\alpha$ and that $\pi_{\pi_1(B),\pi_1(A)}(\langle \pi_1(A), \langle X_\alpha \mid$

[11] i.e., every bounded subset of $\mathbb{P}$ has a least upper bound.
[12] $f : \mathbb{Q} \to \mathbb{P}$ is called monotone if $q_1 \leq_{\mathbb{Q}} q_2 \Rightarrow f(q_1) \leq_{\mathbb{P}} f(q_2)$.

$\alpha \in \pi_1(B)\rangle\rangle) = F(A)$. It follows by monotonicity of the functions, and by (†) that

$$\begin{aligned} F(A) &= f_{\pi_1(A)}(\langle \pi_1(A), \langle (A)_\alpha \mid \alpha \in \pi_1(A)\rangle\rangle) \\ &= f_{\pi_1(A)}(\pi_{\pi_1(B),\pi_1(A)}(\langle \pi_1(A), \langle X_\alpha \mid \alpha \in \pi_1(B)\rangle\rangle) \\ &\geq_{\mathbb{P}} f_{\pi_1(B)}(\langle \pi_1(A), \langle X_\alpha \mid \alpha \in \pi_1(B)\rangle\rangle) \\ &\geq_{\mathbb{P}} f_{\pi_1(B)}(\langle \pi_1(B), \langle (B)_\alpha \mid \alpha \in \pi_1(B)\rangle\rangle = F(B) \end{aligned}$$

To see it is cofinal, let $p \in \mathbb{P}$ be any element, fix any $X \in U$, since f_X is cofinal, there is $Z \in U \restriction X$ and $\langle A_\alpha \mid \alpha \in X\rangle \in \prod_{\alpha\in X} V_\alpha$ such that $f_X(\langle Z, \langle A_\alpha \mid \alpha \in X\rangle\rangle) \geq_{\mathbb{P}} p$. Consider $A = \bigcup_{\alpha\in Z}\{\alpha\} \times A_\alpha$. Then

$$\begin{aligned} F(A) &= f_Z(\langle Z, \langle A_\alpha \mid \alpha \in Z\rangle\rangle) \\ &= f_Z(\pi_{X,Z}(\langle Z, \langle A_\alpha \mid \alpha \in X\rangle\rangle)) \\ &\geq_{\mathbb{P}} f_X(\langle Z, \langle A_\alpha \mid \alpha \in X\rangle\rangle \geq_{\mathbb{P}} p \end{aligned}$$

□

Lemma 2.8. *Suppose that $\mathbb{P}$ is complete and for each $X \in U$, $\mathcal{X}_X \subseteq U \times \prod_{\alpha\in X} V_\alpha$ is such that:*

1. *$\mathcal{X}_X$ is a cofinal subset of $U \times \prod_{\alpha\in X} V_\alpha$.*
2. *$f_X : \mathcal{X}_X \to \mathbb{P}$ is monotone cofinal.*
3. *whenever $Y \subseteq X$, $\pi_{X,Y}(\mathcal{X}_X) \subseteq \mathcal{X}_Y$ and $f_Y(\pi_{X,Y}(C)) \geq_{\mathbb{P}} f_X(C)$*

Then $\mathbb{P}$ is uniformly below $\mathcal{B}(U, \langle B_\alpha \mid \alpha < \lambda\rangle)$.

Proof. Let us define coherent functions $f^*_X : U \times \prod_{\alpha\in X} V_\alpha \to \mathbb{P}$ by

$$f^*_X(A) = \sup\{f_X(B) \mid A \subseteq B \in \mathcal{X}_X\}.$$

Note that if $B' \in \mathcal{X}_X$ is such that $B' \subseteq A$, then the set $\{f_X(B) \mid A \subseteq B \in \mathcal{X}_X\}$ is bound in $\mathbb{P}$ by $f_X(B')$ (as f_X is monotone). Hence $f^*_X(A)$ is well defined by completeness of $\mathbb{P}$. Since f_X is monotone cofinal, f^*_X is monotone cofinal. To see (†), suppose that $Y \subseteq X$, and $C \in U \times \prod_{\alpha<\lambda} V_\alpha$, then for every $C \subseteq B \in \mathcal{X}_X$, by (3), $\pi_{X,Y}(C) \subseteq \pi_{X,Y}(B) \in \mathcal{X}_Y$ and $f_X(B) \leq f_Y(\pi_{X,Y}(B))$. If follows that $f^*_X(C) \leq f^*_Y(\pi_{X,Y}(C))$. □

Corollary 2.9. *Let U be an ultrafilter on $\lambda \geq \omega$ and that each V_α is a δ_α-complete ultrafilter on some $\delta_\alpha > \alpha$. If $\mathbb{P} \leq_T V_\alpha$ for every $\alpha < \lambda$, then $\mathbb{P}^\lambda \leq_T \sum_U V_\alpha$.*

Proof. By moving to the Boolean completion of $\mathbb{P}$, we may assume that $\mathbb{P}$ is complete (see e.g [11]). We fix for every $\alpha < \lambda$, $f_\alpha : V_\alpha \to \mathbb{P}$ monotone and cofinal. For every $X \in U$, we define a cofinal set $\mathcal{X}_X \subseteq U \times \prod_{\alpha\in X} V_\alpha$ consisting of all the elements $\langle Z, \langle A_\alpha \mid \alpha \in X\rangle\rangle$ such that for every $\alpha < \beta$ in X, $f_\alpha(A_\alpha) \leq_{\mathbb{P}} f_\beta(A_\beta)$.

Claim 2.10. *$\mathcal{X}_X$ is cofinal in $U \times \prod_{\alpha\in X} V_\alpha$*

Proof of claim. Let $\langle Z, \langle B_\alpha \mid \alpha \in X\rangle\rangle$. Set $B_0 = A_0$, and recursively suppose that $\beta \in X$ and A_α was defined for all $\alpha \in X \cap \beta$. Since f_α is cofinal, there is $C_\alpha \in V_\beta$ such that $f_\alpha(A_\alpha) \leq_{\mathbb{P}} f_\beta(C_\alpha)$. The $A_\beta := B_\beta \cap \bigcap_{\alpha<\beta} C_\alpha$ is in V_β by δ_β-completeness. $A_\beta \in V_\beta$. By monotonicity of f_β, for every $\alpha < \beta$, $f_\alpha(A_\alpha) \leq_{\mathbb{P}} f_\beta(A_\beta)$. □

Note that $\pi_{X,Y}(\mathcal{X}_X) \subseteq \mathcal{X}_Y$. Define $f_X : \mathcal{X}_X \to \mathbb{P}^\lambda$ by

$$f_X(\langle Z, \langle A_\alpha \mid \alpha \in X\rangle\rangle) = \langle f_{X(\alpha)}(A_{X(\alpha)}) \mid \alpha < \lambda\rangle.$$

Let $\langle Z, \langle A_\alpha \mid \alpha \in X\rangle\rangle \in \mathcal{X}_X$, and let $Y \subseteq X$, then $Y(\alpha) \geq X(\alpha)$. Hence, by definition of $\mathcal{X}_X$, $f_{X(\alpha)}(A_{X(\alpha)}) \leq_{\mathbb{P}} f_{Y(\alpha)}(A_{Y(\alpha)})$. We conclude that

$$\begin{aligned} f_X(\langle Z, \langle A_\alpha \mid \alpha \in X\rangle\rangle) &= \langle f_{X(\alpha)}(A_{X(\alpha)}) \mid \alpha < \lambda\rangle \\ &\leq_{\mathbb{P}^\lambda} \langle f_{Y(\alpha)}(A_{Y(\alpha)}) \mid \alpha < \lambda\rangle \\ &= f_Y(\langle Z, \langle A_\alpha \mid \alpha \in Y\rangle\rangle) = f_Y(\pi_{X,Y}(\langle Z, \langle A_\alpha \mid \alpha \in X\rangle\rangle)). \end{aligned}$$

By Lemma 2.8, $\mathbb{P}^\lambda$ is uniformly below $\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda\rangle)$ and by Theorem 2.7, $\mathbb{P}^\lambda \leq_T \sum_U V_\alpha$. □

In particular, if U_α is Tukey-top for a set of α's in U, then $\sum_U U_\alpha$ is Tukey top.

It is unclear whether every $\mathbb{P}$ which is a Tukey lower bound for $\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda\rangle)$ is Tukey below $\sum_U V_\alpha$. Let us give a few common configurations of the Tukey relation among the ultrafilters V_α in which $\sum_U V_\alpha$ is the greatest lower bound in the usual sense. Let us denote that by $\sum_U V_\alpha = \inf(\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda\rangle))$.

The following is a straightforward corollary from Theorem 2.7:

Corollary 2.11. *Let $X_0 \in U$, then $\sum_U V_\alpha \equiv_T U \times \prod_{\alpha \in X_0} V_\alpha$ if and only if $U \times \prod_{\alpha \in X_0} V_\alpha$ is uniformly below $\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda\rangle)$. In that case $\sum_U V_\alpha = \inf(\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda\rangle)$.*

The following corollary follows from Theorem 2.4.

Corollary 2.12. *Suppose that U, V_n are ultrafilters on ω, such that on a set $X_0 \in U$, for every $n \leq m \in X_0$, $V_n \leq_T V_m$. Then*

$$U \times \prod_{n \in X_0} V_n \equiv_T \sum_U V_n = \inf(\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda\rangle))$$

The second case in which $\sum_U V_\alpha$ turns out to be the greatest lower bound is described in the following Lemma:

Lemma 2.13. *Suppose that there is a set $X_0 \in U$ such that for every $\alpha < \beta \in X_0$, V_α is a κ-complete ultrafilter such that $V_\alpha \cdot V_\alpha \equiv_T V_\alpha >_T V_\beta$. Then $\sum_U V_\alpha = \inf(\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda\rangle))$ is a strict greatest lower bound.*

Proof. First note that for every $Y \subseteq X$, by the assumptions,

$$V_{\min(Y)} \leq_T \prod_{m \in Y} V_m \leq_T \prod_{m \in Y} V_{\min(Y)} \equiv_T V_{\min(Y)} \cdot V_{\min(Y)} \equiv_T V_{\min(Y)}.$$

Therefore, if $\mathbb{P} \leq_T B$ for every $B \in \mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda\rangle)$ then $\mathbb{P} \leq_T V_\alpha$ for every $\alpha \in X$. By corollary 2.9, it follows than that $\mathbb{P} \leq_T \sum_U V_\alpha$. Moreover, $\sum_U V_\alpha$ is strictly below $\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda\rangle)$, since if $\beta < \lambda$, then $\sum_U V_\alpha \leq_T V_{\beta+1} <_T V_\beta$. □

Our next goal is to prove that the assumptions of Lemma 2.13 are consistent. To do that, we will need a theorem of Raghavan and Todorcevic from [16] regarding the canonization of cofinal maps from basically generated ultrafilters. Basically generated ultrafilters were introduced by Dobrinen and Todorcevic [9], we say that U is *basically generated* if there is a cofinal set $\mathcal{B} \subseteq U$ closed under intersections, such that for every sequence $\langle b_n \mid n < \omega\rangle \subseteq \mathcal{B}$ which converges[13] to an element of $\mathcal{B}$, there is $I \in [\omega]^\omega$ such that $\bigcap_{i \in I} A_i \in U$. Dobrinen and Todorcevic proved that p-point ultrafilter U is basically generated and that the class of basically generated ultrafilters is closed under sums ([9, Thm. 14 & 16]).

[13]A sequence $\langle A_n \mid n < \omega\rangle$ of subsets ω is said to converge to A if for every $n < \omega$ there is $N < \omega$ such that for every $m \geq N$, $A_m \cap n = A \cap n$.

Theorem 2.14 (Raghavan-Todorcevic)**.** *Let U be a basically generated ultrafilter and V be any ultrafilter such that $V \leq_T U$. Then there is $P \subseteq \text{FIN}$ such that:*

1. $\forall t, s \in P,\ t \subseteq s \Rightarrow t = s$.
2. *V is Rudin-Keisler below $U(P)$, namely, there is $f : P \to \omega$ such that $V = \{X \subseteq \omega \mid f^{-1}[X] \in U(P)\}$.*
3. $U(P) \equiv_T V$.

Where is the filter $U(P) = \{A \subseteq P \mid \exists a \in U.[a]^{<\omega} \subseteq A\}$.

The forcing notion $P(\omega)/\text{fin}$ consists of infinite sets, ordered by inclusion up to a finite set. Namely, $X \leq^* Y$ if $X \setminus Y$ is finite. In the next proposition, we consider the forcing notion $\mathbb{P} = \prod_{n<\omega} P(\omega)/\text{fin}$, where elements of the product have full support. For more information regarding forcing we refer the reader to [14].

The following items summarize the properties of $\mathbb{P}$ which we will need:

- $\mathbb{P}$ is σ-closed, and therefore does not add reals, and ω_1 is preserved.
- The projection to the n^{th} coordinate projects $\mathbb{P}$ to $P(\omega)/\text{fin}$[14].
- If $G \subseteq \mathbb{P}$ is generic over V, then $U_n := \overline{\pi_n(G)} = \{X \in P(\omega) \mid \exists f \in G.f(n) \leq^* X\}$ is an ultrafilter over ω in $V[G]$.
- Each U_n is a selective ultrafilter and $U_n \notin V[\langle U_m \mid m \in \omega \setminus \{n\}\rangle]$.

Proposition 2.15. *Let $\mathbb{P}$ be a full support product of ω-copies of $P(\omega)/\text{fin}$. Let $G \subseteq \mathbb{P}$ be generic over V. Then in $V[G]$ there is a sequence of ultrafilters V_n, such that $V_0 >_T V_1 >_T V_2 \ldots$ and $V_n \cdot V_n \equiv V_n$.*

Proof. For each $n < \omega$, U_n is a selective ultrafilter and therefore by Dobrinen and Todorcevic [9], $U_n \cdot U_n \equiv_T U_n \equiv_T (U_n)^\omega$. For every $n < \omega$, define[15]

$$V_n = \sum_{U_0} (U_{n+1} \cdot U_{n+2} \cdot \ldots \cdot U_{n+m})_{0<m<\omega}$$

Note that each V_n is basically generated as the product and sum of such.

Lemma 2.16.

1. $V_n \equiv_T U_0 \times \prod_{n<m<\omega} U_m$.
2. $V_n \cdot V_n \equiv_T V_n$.
3. $V_0 >_T V_1 >_T V_2 \ldots$.

Proof of Lemma. For (1), we note that for each $0 < m$, the ultrafilters $U_{n+1} \cdot \ldots \cdot U_{n+m} \leq_T U_{n+1} \cdot \ldots \cdot U_{n+m+1}$. Hence by Theorem 2.4,

$$V_n \equiv_T U_0 \times \prod_{0<m<\omega} U_{n+1} \cdot \ldots \cdot U_{n+m}$$

By Milovich's Theorem 0.1, and by our assumptions, for each n, m

$$U_{n+1} \cdot \ldots \cdot U_{n+m} \equiv_T U_{n+1} \times (U_{n+2} \cdot U_{n+2}) \times \ldots \times (U_{n+m} \cdot U_{n+m}) \equiv_T U_{n+1} \times \ldots \times U_{n+m}.$$

[14]A function from $f : \mathbb{P} \to \mathbb{Q}$ is called a projection of forcing notions if f is order-preserving, $\text{rng}(f)$ is dense in $\mathbb{Q}$, and for every $p \in \mathbb{P}$ and $q \leq_{\mathbb{Q}} p$, there is $p' \leq_{\mathbb{P}} p$ such that $f(p') \leq_{\mathbb{Q}} q$.

[15]We thank Gabe Goldberg for pointing out this definition of V_n.

Hence

$$\begin{aligned} V_n &\equiv_T U_0 \times \prod_{0<m<\omega} U_{n+1} \times \ldots \times U_{n+m} \\ &\equiv_T U_0 \times \prod_{0<m<\omega} (U_{n+m})^\omega \\ &\equiv_T U_0 \times \prod_{0<m<\omega} U_{n+m} \cdot U_{n+m} \equiv_T U_0 \times \prod_{0<m<\omega} U_{n+m} \end{aligned}$$

For (2), we use (1). For each $n < \omega$,

$$\begin{aligned} V_n \cdot V_n &\equiv_T (V_n)^\omega \\ &\equiv_T (U_0 \times \prod_{0<m<\omega} U_{n+m})^\omega \\ &\equiv_T (U_0)^\omega \times \prod_{0<m<\omega} (U_{n+m})^\omega \\ &\equiv_T U_0 \times \prod_{0<m<\omega} U_{n+m} \equiv_T V_n \end{aligned}$$

To see (3), note that from (1), we have $V_0 \geq_T V_1 \geq_T V_2$..... Suppose toward a contradiction that $V_n \equiv_T V_{n+1}$ for some n. Then $U_{n+1} \leq_T V_{n+1}$. Note that

$$V_{n+1} \in V[U_0, \langle U_m \mid n+1 < m < \omega\rangle].$$

By mutual genericity $U_{n+1} \notin V[U_0, \langle U_m \mid n+1 < m < \omega\rangle]$. Since V_{n+1} is basically generated, Theorem 2.14 implies that there is $P \subseteq FIN$ such that $U_{n+1} \leq_{RK} V_n(P)$. Note that since $\mathbb{P}$ is σ-closed, $P \in V$ and therefore $V_n(P) \in V[U_0, \langle U_m \mid n+1 < m < \omega\rangle]$. Also the Rudin-Keisler projection f such that $f_*(V_n(P)) = U_{n+1}$ is in the ground model and therefore $U_{n+1} \in V[U_0, \langle U_m \mid n+1 < m < \omega\rangle]$, contradiction. □

□

It follows that $\sum_{U_0} V_n = \inf(\mathcal{B}(U_0, \langle V_n \mid 0 < n < \omega\rangle))$ is a strict greatest lower bound. Let us also prove that $U_0 <_T \sum_U V_n$. We need the following standard fact.

Fact 2.17. Suppose that $\sum_U V_n = \sum_U V'_n$ then $\{n < \omega \mid V_n = V'_n\} \in U$

Proof. Just otherwise, $Y = \{n < \omega \mid V_n \neq V'_n\} \in U$, in which case, for every $n \in Y$ take $X_n \in V_n$ such that $X_n^c \in V'_n$. Then $A = \bigcup_{n\in Y}\{n\} \times X_n \in \sum_U V_n$, while $A' = \bigcup_{n\in Y}\{n\} \times X_n^c \in \sum_U V'_n$. However $A \cap A' = \emptyset$ which contradicts $\sum_U V_n = \sum_U V'_n$. □

Proposition 2.18. $U_0 <_T \sum_{U_0} V_n$

Proof. Otherwise, there would have been a continuous cofinal map $f : U_0 \to \sum_{U_0} V_n$. Since U_0 is a selective ultrafilter, by Todorcevic [16], if $V \leq_T U_0$, then there is $\alpha < \omega_1$ such that $V =_{RK} U_0^\alpha$ for some $\alpha < \omega_1$. It follows that $\sum_{U_0} V_n \equiv_{RK} U_0^\alpha$ for some $\alpha < \omega_1$. If $\alpha > 1$, then $U_0^\alpha = \sum_{U_0} U_0^{\alpha_n}$ for some $\alpha_n < \alpha$ (The α_n's might be constant). It follows that $Y = \{n < \omega \mid V_n =_{RK} U_0^{\alpha_n}\} \in U_0$. Since for any $\beta < \omega_1$, $U_0^\beta \in V[U_0]$, for any $0 < n \in Y$, we conclude that $V_n \in V[U_0]$ and in particular $U_1 \in V[U_0]$, contradicting the mutual genericity. If $\alpha = 1$, then $U_0 =_{RK} \sum_{U_0} V_n$ which implies that $\sum_{U_0} V_n$ is a p-point, contradicting Fact 1.3.

□

3. Commutativity of cofinal types

In this section, we provide some information regarding Main Question 2: whether every two ultrafilters U, V satisfy $U \cdot V \equiv_T V \cdot U$. Let us start with a few consequences of commutativity:

Proposition 3.1. *Suppose that U, W are ultrafilters on ω satisfying $U \cdot W \equiv_T W \cdot U$ if and only if $U \cdot W \equiv_T U^\omega \times W^\omega$ and $W \cdot U \equiv_T W^\omega \times U^\omega$.*

Proof. The implication from right to left follows easily since $U^\omega \times W^\omega \equiv_T W^\omega \times U^\omega$. For the other direction, recall that by Theorem 0.1 that $U \cdot W \equiv_T U \times W^\omega$ and $W \cdot U \equiv_T W \times U^\omega$. Hence, if $U \cdot W \equiv_T W \cdot U$ then $W \times U^\omega \equiv_T U \times W^\omega$ from which it follows that $W \times U^\omega \equiv_T U \times W^\omega \equiv_T U^\omega \times W^\omega$. □

We say that a class C of ultrafilters on ω is a *commutative class* if for every $U, W \in C$, $U \cdot W \equiv_T W \cdot U$. The previous proposition says that a certain class of ultrafilter is commutative, the reason must be that inside that class, $U \cdot W$ has a formula which is *symmetric* in U, W. This is formally expressed in (1) of the following proposition.

Proposition 3.2. *Let C be a class of ultrafilters.*

1. *C is commutative if and only if there is $f : \beta\omega \times \beta\omega \to P(\mathbb{DO}(\mathfrak{c}))$ [16] such that:*
   - (a) *f is symmetric: $f(U, W) = f(W, U)$.*
   - (b) *For all $U, W \in C$, $U \cdot W \equiv_T \prod f(U, W)$.*
2. *If C is a commutative class then so is $\{U_1 \cdot U_2 ... \cdot U_n \mid U_1, ..., U_n \in C\}$*
3. *Let D be a cofinal type, denote by $\mathcal{E}_D = \{U \mid U \cdot U \equiv_T U \times D\}$. Then $\mathcal{E}_D$ is a commutative class.*
4. *Suppose that $U \in \mathcal{E}_D$ and $U \leq_T W$ and U commutes with W, then $W \in \mathcal{E}_D$.*

Proof. To see (1), for the implication from left to right, set $f(U, W) = \{U^\omega, W^\omega\}$. Then $f(U, W) = f(W, U)$ and by the previous proposition, for $U, W \in C$, $U \cdot W \equiv_T U^\omega \times W^\omega = \prod f(U, W)$. For the other direction, given f satisfying (a),(b), we have that

$$U \cdot W \equiv_T \prod f(U, W) = \prod f(W, U) \equiv_T W \cdot U$$

for all $U, W \in C$. Hence C is a commutative class.

(2) follows by the associativity $U \cdot (W \cdot Z) \equiv_{RK} (U \cdot W) \cdot Z$ for any three ultrafilters U, W, Z. For (3), take any $U, V \in \mathcal{E}_D$. By Theorem 0.1

$$V \cdot U \equiv_T V \times U^\omega \equiv_T V \times U \cdot U \equiv_T V \times U \times D.$$

The formula above is symmetric for V, U, and by (1), $\mathcal{E}_D$ is commutative.

To see (4), First note that $U \leq_T W$ and

$$\begin{aligned} W \times D &\equiv_T W \times U \times D \equiv_T W \times U \cdot U \\ &\leq_T W \times W \cdot W \equiv_T W \cdot W \\ &\equiv_T U \times W \cdot W \equiv_T U \cdot W \\ &\equiv_T W \cdot U \equiv_T W \times U \times D \equiv_T W \times D \end{aligned}$$

It follows that $W \cdot W \equiv_T W \times D$. □

Example 3.3. Consider $\mathcal{E}_{\{0\}}, \mathcal{E}_\omega$, and $\mathcal{E}_{\omega^\omega}$. Is it easy to see that

$$\mathcal{E}_{\{0\}} = \mathcal{E}_\omega = \{U \mid U \cdot U \equiv_T U\}.$$

[16] We denote by $\mathbb{DO}(\mathfrak{c})$ the class of directed orders of size at most $\mathfrak{c}$.

Dobrinen and Todorcevic [9] showed that if U is a rapid p-point then $U \cdot U \equiv_T U$, namely $U \in \mathcal{E}_0$. Also, Milovich [15] showed that if W is a p-point, then $W \cdot W \equiv_T W \times \omega^\omega$, namely $\mathcal{E}_{\omega^\omega}$ includes all p-points.

Claim 3.4. $\mathcal{E}_{\{0\}} \subseteq \mathcal{E}_{\omega^\omega}$.

Proof. Just note that $\omega^\omega \leq_T U \cdot U$ for every uniform ultrafilter U and therefore if $U \cdot U \equiv_T U$ then $U \equiv_T U \times \omega^\omega$ and in particular $U \cdot U \equiv_T U \times \omega^\omega$.
$U \cdot W$ $W \cdot W \equiv_T W \times \omega^\omega$. $U \cdot U \equiv_T U \times I^\omega$. $U \cdot W \equiv_T U \times W \times \omega^\omega$ □

Let us turn to the main Theorem of this section:

Theorem 3.5. *$\mathcal{E}_{\omega^\omega}$ is closed under Fubini sums.*

Proof. Suppose that $\{W, W_0, W_1, ...\} \subseteq \mathcal{E}_{\omega^\omega}$. We need to prove that

$$\sum_W W_n \cdot \sum_W W_n \equiv_T \sum_W W_n \times \omega^\omega.$$

Note that $\sum_W W_n \geq_T \omega^\omega$, so we will end up getting $\sum_W W_n \cdot \sum_W W_n \equiv_T \sum_W W_n$. It is not hard to see that

$$\sum_W W_n \cdot \sum_W W_n \geq_T \sum_W W_n \times \omega^\omega.$$

For the other direction, recall that $\sum_W W_n \cdot \sum_W W_n \equiv_T (\sum_W W_n)^\omega$. Let us prove that $(\sum_W W_n)^\omega$ is a uniformly below $\mathcal{B}(W, \langle W_n \mid n < \omega\rangle)$. Since $(\sum_W W_n)^\omega$ is complete, Theorem 2.7 can then be applied that get $\sum_W W_n \geq_T (\sum_W W_n)^\omega$.

Let $f : W \times \omega^\omega \to W^\omega$ and $f_n : W_n \times \omega^\omega \to W_n^\omega$ be monotone and cofinal maps, which exists by the assumption that $W, W_n \in \mathcal{E}_{\omega^\omega}$. We need to define a sequence of monotone and cofinal maps $\langle g_X : W \times \prod_{n \in X} W_n \to (\sum_W W_n)^\omega \mid X \in W\rangle$ such that (†) of definition 2.6 holds.

Let $\langle B, \langle A_n \mid n \in X\rangle\rangle \in W \times \prod_{n \in X} W_n$, by lemma 2.8, we may restrict ourselves to sequences satisfying that

$$(\star) \quad \text{for every } n_1, n_2 \in X,\ n_1 < n_2 \Rightarrow \min(A_{n_1}) < \min(A_{n_2}).$$

The first step is to produce ω-many functions in ω^ω which are going to be the inputs of f_n's. Fix a partition of ω i.e. $\omega = \bigcup_{l<\omega} Z_l$ such that the Z_l's are pairwise disjoint and infinite. Recall that for a set C of natural numbers, $C(r)$ denotes the r^{th} element of C in its increasing enumeration. Define $\varphi_{X,i}$ by induction on i. For $i = 0$, $\varphi_{X,0}(k) = X(Z_0(k))$. Inductively, $\varphi_{X,i+1}(k) = \max(\varphi_{X,i}(k), X(Z_{i+1}(k)))$.

Claim 3.6. *If $Y \subseteq X$, then for every i, $\varphi_{X,i} \leq \varphi_{Y,i}$.*

Proof. Clearly, for every $m < \omega$, $X(m) \leq Y(m)$. So by definition, $\varphi_{X,0} \leq \varphi_{Y,0}$. Suppose this was true for i, and let $k < \omega$, then by the induction hypothesis and our first observation,

$$\max(\varphi_{X,i}(k), X(Z_{i+1}(k))) \leq \max(\varphi_{Y,i}(k), Y(Z_{i+1}(k))).$$

□

The i's function we will use is $h^i_{\langle A_n \mid n \in X\rangle}(k) = \min(A_{\varphi_{X,i}(k)})$. This is well defined as by the definition of $\varphi_{X,i}$, $\varphi_{X,i}(k) \in X$. If $Y \subseteq X$, then by the claim $\varphi_{X,i}(k) \leq \varphi_{Y,i}(k)$, and by $(\star)$, $h^i_{\langle A_n \mid n \in X\rangle}(k) \leq h^i_{\langle A_n \mid n \in Y\rangle}(k)$. Now define $g_X(\langle B, \langle A_k \mid k \in X\rangle\rangle)$ by

$$\langle \pi_X(\langle f(B, h^0_{\langle A_k \mid k \in X\rangle})_m, \langle f_n(A_n, h^{n+1}_{\langle A_k \mid k \in X\rangle})_m \mid n \in X\rangle) \mid m < \omega\rangle$$

The above seemingly complicated definition is nothing but the composition of the following quite natural monotone cofinal maps:

$$\begin{aligned} W\times\prod_{n\in X}W_n &\overset{(id,\langle h^i_*\mid i<\omega\rangle)}{\longrightarrow} W\times(\prod_{n\in X}W_n)\times(\omega^\omega)^\omega\\ &\longrightarrow (W\times\omega^\omega)\times\prod_{n\in X}(W_n\times\omega^\omega)\\ &\overset{(f,\langle f_n\mid n\in X\rangle)}{\longrightarrow} W^\omega\times\prod_{n\in X}W_n^\omega\\ &\longrightarrow (W\times\prod_{n\in X}W_n)^\omega\overset{\pi_X^\omega}{\longrightarrow}(\sum_W W_n)^\omega\end{aligned}$$

So g_X is clearly monotone cofinal as the composition of such functions. To see (†), let $Y\subseteq X$, we ensured that $h^i_{\langle A_n\mid n\in X\rangle}\leq h^i_{\langle A_n\mid n\in Y\rangle}$. Since f,f_n are monotone functions, for any $m<\omega$,

$$B_{Y,m}:=f(B,h^0_{\langle A_k\mid k\in Y\rangle})_m\subseteq f(B,h^0_{\langle A_k\mid k\in X\rangle})_m=:B_{X,m}$$

and for any $n\in Y$,

$$B^n_{Y,m}:=f_n(A_n,h^{n+1}_{\langle A_k\mid k\in Y\rangle})_m\subseteq f_n(A_n,h^{n+1}_{\langle A_k\mid k\in X\rangle})_m=B^n_{X,m}.$$

By definition of π_X and π_Y:

$$\begin{aligned}\pi_Y(\langle B_{Y,m},\langle B^n_{Y,m}\mid n\in Y\rangle\rangle)&=\bigcup_{n\in Y\cap B_{Y,m}}\{n\}\times B^n_{Y,m}\\ &\subseteq\bigcup_{n\in X\cap B_{X,m}}\{n\}\times B^n_{X,m}=\pi_X(\langle B_{X,m},\langle B^n_{X,m}\mid n\in X\rangle\rangle)\end{aligned}$$

By definition of g_Y,g_X and $\pi_{X,Y}$, for all $m<\omega$:

$$\begin{aligned}g_Y(\pi_{X,Y}(\langle B,\langle A_n\mid n\in X\rangle\rangle))_m&=g_Y(\langle B,\langle A_n\mid n\in Y\rangle\rangle)_m\\ &=\pi_Y(\langle B_{Y,m},\langle B^n_{Y,m}\mid n\in Y\rangle\rangle)\\ &\subseteq\pi_X(\langle B_{X,m},\langle B^n_{X,m}\mid n\in X\rangle\rangle)=g_X(\langle B,\langle A_n\mid n\in X\rangle\rangle)_m\end{aligned}$$

Hence, $g_Y(\pi_{X,Y}(\langle B,\langle A_n\mid n\in X\rangle\rangle))\geq g_X(\langle B,\langle A_n\mid n\in X\rangle\rangle)$. □

The main question remaining from this paper is still, whether the commutativity of cofinal types is true in general. Let us formulate two related questions which relate to the results of this paper:

Question 3.7. Is is consistent that $\mathcal{E}_{\omega^\omega}$ includes all non principal ultrafilters?

Question 3.8. Is the class of basically generated ultrafilters a subclass of $\mathcal{E}_{\omega^\omega}$?

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