COMMUTATIVITY OF COFINAL TYPES

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Abstract. We developed the theory of deterministic ideals and present a systematic study of the pseudo-intersection property with respect to an ideal introduced in [3]. We apply this theory to prove that for any two ultrafilters U, V over ω, $U \cdot V \equiv_T V \cdot U$. This is in sharp contrast to the Rudin-Keisler ordering. Our theory applies to the study of the Tukey types of general sums of ultrafilters, which, as evidenced by the results of this paper, can be quite complex. In the third part of this paper, we apply our results to study the class of ultrafilters Tukey above ω^ω. Specifically, we prove that ultrafilters without the I-p.i.p are always above I^ω and in particular non-p-points are Tukey above ω^ω. Finally, we introduce the hierarchy of α-almost rapid ultrafilters. We prove that it is consistent for them to form a strictly wider class than the rapid ultrafilters, and give an example of a non-rapid p-point ultrafilter which is Tukey above ω^ω. This addresses and answers several questions from [2, 3, 14, 28].

0. Introduction

The Tukey order stands out as one of the most studied orders of ultrafilters [28, 14, 25, 10, 32, 2]. Its origins lie in the examination of Moore-Smith convergence, and it holds particular significance in unraveling the cofinal structure of the partial order $(U, \supseteq)$ of an ultrafilter. Formally, given two posets, $(P, \leq_P)$ and $(Q, \leq_Q)$ we say that $(P, \leq_P) \leq_T (Q, \leq_Q)$ if there is map $f : Q \to P$, which is cofinal, namely, $f''\mathcal{B}$ is cofinal in P whenever $\mathcal{B} \subseteq Q$ is cofinal. Schmidt [33] observed that this is equivalent to having a map $f : P \to Q$, which is unbounded, namely, $f''\mathcal{A}$ is unbounded in Q whenever $\mathcal{A} \subseteq P$ is unbounded in P. We say that P and Q are *Tukey equivalent*, and write $P \equiv_T Q$, if $P \leq_T Q$ and $Q \leq_T P$; the equivalence class $[P]_T$ is called the *Tukey type* or *cofinal type* of P.

The scope of the study of cofinal types of ultrafilters covers several long-standing open problems such as:

- Isbell's problem[19]: Is it provable within ZFC that a non-Tukey-top ultrafilter[1] on ω exists?

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[1] A Tukey-top ultrafilter is an ultrafilter which is Tukey equivalent to the poset $([\mathfrak{c}]^{<\omega}, \subseteq)$. Isbell [19] constructed such ultrafilters in ZFC.

- Kunen's Problem: Is it consistent that $\mathfrak{u}_{\aleph_1} < 2^{\aleph_1}$? Namely, is it consistent to have a set $\mathcal{B} \subseteq P(\omega_1)$ of cardinality less than $2^{\aleph_1}$ which generates an ultrafilter?

The Tukey order is also related to the Katovich problem. A systematic study of the Tukey order on ultrafilter over ω, traces back to Isbell [19], later to Milovich [28] and Dobrinen and Todorcevic [14]. Lately, Benhamou and Dobrinen [2] extended this study to ultrafilters on cardinals greater than ω. Over measurable cardinals, the Tukey order is connected to recent developments revolving the so-called Galvin property, studied by Abraham, Benhamou, Garti, Goldberg, Gitik, Hayut, Magidor, Poveda, Shelah and others [1, 15, 16, 6, 4, 5, 8, 7, 17, 9]; the Galvin property in one of its forms is equivalent to being Tukey-top as shown essentially by Isbell (in different terminology). Moreover, being Tukey-top in the restricted class of κ-complete ultrafilters takes the usual studied forms of the Galvin property.

In this paper, we address the problem of commutativity of the Tukey types of Fubini products of ultrafilters U, V over ω (Definition 1.1), denoted by $U \cdot V$. This problem was suggested in [3], and was already partially addressed:

- Dobrinen and Todorcevic [14] proved that if U, V are rapid p-points then $U \cdot V \equiv_T V \cdot U$.
- Milovich [29] extended this result to prove that if U, V are just p-points, then $U \cdot V \equiv_T V \cdot U$.
- Benhamou and Dobrinen proved later that if U, V are κ-complete ultrafilters over a measurable cardinal κ then $U \cdot V \equiv_T V \cdot U$.

The main result of this paper is:

Theorem. *For any ultrafilters $U_1, U_2, ..., U_n$ on ω, and any permutation $\sigma : \{1, ..., n\} \to \{1, ..., n\}$, $U_1 \cdot U_2 \cdot ... \cdot U_n \equiv_T U_{\sigma(1)} \cdot ... \cdot U_{\sigma(n)}$.*

To do this we will, in fact, prove that

$$(*) \quad U_1 \cdot U_2 \cdot ... \cdot U_n \equiv_T U_1 \times U_2 \times ... \times U_n \times (U_1 \cap ... \cap U_n)^\omega$$

from which the commutativity follows straightforwardly.

Our main result stands in sharp contrast to the Rudin-Keisler ordering which is known not to be commutative with respect to Fubini product[2]. On measurable cardinals, the situation is even more dramatic, due to a theorem of Solovey (see [20, Thm. 5.7]) if U, W are κ-complete ultrafilters on κ the $U \cdot W \equiv_T W \cdot U$ if and only if $W \equiv_{RK} U^n$ for some n or vise versa. Recently, Goldberg [18] examined situations of commutativity with respect to several product operations on countably complete ultrafilters.

As a corollary of our main theorem, we conclude the following Tukey-structural results:

[2]For example if U, W are non-isomorphic Ramsey ultrafilters then $U \cdot W \not\equiv_{RK} W \cdot U$. Just otherwise, by a theorem of Rudin (see for example [20, Thm. 5.5], U, W should be Rudin-Frolík (and therefore Rudin-Keisler) comparable, contradicting the RK-minimality of Ramsey ultrafilters.

Corollary. *The class of ultrafilters U such that $U \cdot U \equiv_T U$ is upward closed with respect to the Tukey order.*

The main idea that is used in the proof of the main result, as suggested by the equivalence $(*)$, is to analyze the cofinal types of ideals and filters connected to a given ultrafilter U. More specifically, we will exploit the idea of the *pseudo-intersection property with respect to I* (Definition 2.2) which was introduced in [3] and was used to prove that Milliken-Taylor ultrafilters and generic ultrafilters[3] for $P(\omega^\alpha)/\text{fin}^{\otimes\alpha}$ satisfy $U \cdot U \equiv_T U$. In §2, we provide a comprehensive study of this property. The main result of this section is

Theorem. *Suppose that $\mathcal{A}$ is a discrete set of ultrafilters. Then for each $U \in \mathcal{A}$, U has the $(\bigcap \mathcal{A})^*$-p.i.p.*

The property of ideals which will be in frequent use, is the property of being *deterministic* (Definition 2.22). This property guarantees that whenever $I \subseteq J$, $I \leq_T J$.

We also investigate the Tukey type of ultrafilters of the form $\sum_U V_\alpha$. The only general result regarding the Tukey-type of such ultrafilters is due[4] to Dobrinen and Todorcevic [14] where they prove that if U is an ultrafilter on κ, and V_α is a sequence of ultrafilters, then $\sum_U V_\alpha \leq_T U \times \prod_{\alpha<\kappa} V_\alpha$. It turns out that the Tukey class of such ultrafilters is much more complicated and the nice characterization we have for $U \cdot V \equiv_T U \times V^\omega$ is missing in the general case. It is not hard to see that the ultrafilter $\sum_U V_\alpha$ is Tukey below each ultrafilter in the set $\mathcal{B}(U, \langle V_\alpha \mid \alpha < \kappa \rangle) = \{U \times \prod_{\alpha \in X} V_\alpha \mid X \in U\}$. We prove that in some sense it is the greatest lower bound:

Theorem. *For complete directed ordered set $\mathbb{P}$, $\mathbb{P}$ is uniformly below* [5] *$\mathcal{B}(U, \langle V_\alpha \mid \alpha < \kappa \rangle)$ if and only if $\sum_U V_\alpha \geq_T \mathbb{P}$.*

By putting more assumptions on the sequence of ultrafilters, we are able to get nicer results. One of them is that under the assumption

$$V_0 \leq_T V_1 \leq_T \dots.$$

in which case we have that $\sum_U V_n \equiv_T U \times \prod_{n<\omega} V_n$. From this special case, we recover D. Milovich's formula $U \cdot V \equiv_T U \times V^\omega$ (See Theorem 1.7). Another set of assumptions we consider is when for each $n < \omega$, $V_n \cdot V_n \equiv_T V_n \geq_T V_{n+1}$. These assumptions imply that $\sum_U V_n$ is a strict greatest least lower bound of $\mathcal{B}(U, \langle V_n \mid n < \omega \rangle)$. This shows that the cofinal type of $\sum_U V_n$ is much more complicated than the cofinal type of $U \cdot V$. We also provide an example (Prop. 1.21) where $U <_T \sum_U V_n < U \times \prod_{n \in X} V_n$ for every $X \in U$.

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[3]For the definition of $I^{\otimes\alpha}$, see the paragraph before Fact 2.11.

[4]Dobrinen and Todorcevic proved in for $\kappa = \omega$, but the proof for a general κ appears in [2].

[5]See Definition 1.11.

Finally, we study the class of ultrafilters U such that $U \geq_T \omega^\omega$. The partial order ω^ω appeared quite a bit in the literature [36, 24, 25] in the context of the Tukey order on Borel ideals. In the context of general ultrafilters on ω, it is known to be the immediate successor of the Tukey type ω [24]

Theorem 0.1 (Louveau-Velickovic). *If I is any ideal such that $I <_T \omega^\omega$ then I is countably generated.*

On the other hand, among analytic ideals, it is known to be minimal [37](See also [25, Thm 6.6]):

Theorem 0.2 (Todorcevic). *Suppose that I is an analytic p-ideal, then either I is countable generated or $I \geq_T \omega^\omega$.*

This was later improved by Solecki and Todorcevic [36, Proposition 4.3] to show that if I is analytic, not locally compact ideal, then $I \geq_T \omega^\omega$. This partial order came up in the work of Milovich who asked [28, Question 4.7] if there is an ultrafilter U over ω such that $(U, \supseteq) \equiv_T \omega^\omega$. We will observe that this was basically answered in [36, Cor. 54]:

Theorem 0.3 (Solecki-Todorcevic). *Suppose that D is an ordered separable metric space such that the predecessors of each element form a compact set, and E is a basic [6] analytic order such that $D \leq_T E$, then D is analytic.*

Later, Dobrinen and Todorcevic contributed a great deal to the understanding of this class, in particular, they proved the following [14, Thm. 35]:

Theorem 0.4 (Dobrinen-Todorcevic). *The following are equivalent for a p-point:*

(1) $U \cdot U \equiv_T U$.
(2) $U \geq_T \omega^\omega$.

They also proved that rapid ultrafilters are above ω^ω and deduced that rapid p-points satisfy $U \cdot U \equiv_T U$. This was lately improved by Benhamou and Dobrinenn [3, Thm 1.18] to the general setup of the I-p.i.p.

Theorem 0.5 (Dobrinen-B.). *Let U be an ultrafilter. Then the following are equivalent:*

(1) $U \cdot U \equiv_T U$.
(2) There is an ideal $I \subseteq U^*$ such that $U \geq_T I^\omega$ and U has the I-p.i.p.

Taking $I = \text{fin}$ reproduces the difficult part from Dobrinen and Todorcevic's result[7]. In §4, we first note that in some sense the assumption that U has the I-p.i.p in the theorem above is not optimal.

Theorem. *Let U be any ultrafilter over a set X. If U does not have the I-p.i.p then $U \geq_T I^X$.*

[6]For the definition of basic see [36, §3].

[7]Indeed, as pointed out by Dobrinen and Todorcevic, it is easy to see that $U \cdot U \geq_T \omega^\omega$, so (1) $\Rightarrow$ (2) is straightforward.

In particular, we get the following corollary:

Corollary. *If U is not a p-point ultrafilter over ω then $U \geq_T \omega^\omega$.*

This last corollary is of the same flavor as Theorems 0.1,0.2. Hence if we drop the p-point assumption, then we also get $U \geq_T \omega^\omega$. Hence, if we are looking for examples for ideals that are not Tukey above ω^ω and are not countably generated, we might as well restrict our attention to non-analytic p-ideals. This is closely related to [14, Question 42].

Inside the class of p-point, it is known to be consistent that there are p-points which are not above ω^ω (see [14]), and the class of rapid p-point is currently the largest subclass of p-points known to be above ω^ω. In the second part of Section 4, we enlarge this class by introducing the notion of α-almost-rapid (Definition 4.12), which is a weakening of rapidness.

Theorem. *Suppose that U is α-almost-rapid, then $U \geq_T \omega^\omega$.*

Finally, we prove that the class of almost rapid ultrafilters is a strict extension of the class of rapid ultrafilters, even among p-points.

Theorem. *Assume CH. Then there is a non-rapid almost-rapid p-point ultrafilter.*

This theorem produces a large class of ultrafilters which are above ω^ω. This paper is organized as follows:

- In §1, we start with some preliminary definitions and known results. The main goal of this section is the investigation of the cofinal type of $\sum_U V_\alpha$.
- In §2, we provide a systematic study of the I-p.i.p and deterministic ideals.
- In §3, we prove our main results.
- In §4 we investigate the class of ultrafilters Tukey above ω^ω.
- In §5 we present some open problems and possible directions.

Notations. $[X]^{<\lambda}$ denotes the set of all subsets of X of cardinality less than λ. Let $\text{fin} = [\omega]^{<\omega}$, and $\text{FIN} = \text{fin} \setminus \{\emptyset\}$. For a collection of sets $(P_i)_{i\in I}$ we let $\prod_{i\in I} P_i = \{f : I \to \bigcup_{i\in I} P_i \mid \forall i,\ f(i) \in P_i\}$. If $P_i = P$ for every i, then $P^I = \prod_{i\in I} P$. Given a set $X \subseteq \omega$, such that $|X| = \alpha \leq \omega$, we denote by $\langle X(\beta) \mid \beta < \alpha\rangle$ be the increasing enumeration of X. Given a function $f : A \to B$, for $X \subseteq A$ we let $f''X = \{f(x) \mid x \in X\}$, for $Y \subseteq B$ we let $f^{-1}Y = \{x \in X \mid f(x) \in Y\}$, and let $\text{rng}(f) = f''A$. Given sets $\{A_i \mid i \in I\}$ we denote by $\biguplus_{i\in I} A_i$ the union of the A_i's when the sets A_i are pairwise disjoint. Two partially ordered set $\mathbb{P}, \mathbb{Q}$ are isomorphic, denoted by $\mathbb{P} \simeq \mathbb{Q}$, if there is a bijection $f : \mathbb{P} \to \mathbb{Q}$ which is order-preserving.

1. On the Cofinal types of Fubini sums of ultrafilters

1.1. **Some basic definitions and facts.** The principal operation we are considering in this paper is the Fubini/tensor sums and products of ultrafilters.

Definition 1.1. Suppose that F is a filter over an infinite set X and for each $x \in X$, G_x is a filter over an infinite set Y_x. We denote by $\sum_F G_x$ the filter over $\bigcup_{x\in X}\{x\} \times Y_x$, defined by

$$A \in \sum_F G_x \text{ if and only if } \{x \in X \mid (A)_x \in G_x\} \in F$$

where $(A)_x = \{y \in Y_x \mid \langle x, y\rangle \in A\}$ is the x^{th}*-fiber* of A. If for every x, $G_x = G$ for some fixed V over a set Y, then $F \cdot G$ is defined as $\sum_F G$, which is a filter over $X \times Y$. F^2 denotes the filter $F \cdot F$ over $X \times X$.

We distinguish here between $F \cdot G$ and $F \times G$ which is the cartesian product of F and G with the order defined pointwise[8].

A filter F on a regular cardinal $\kappa \geq \omega$ is called *uniform*[9] if $J^*_{bd} = \{X \subseteq \kappa \mid \kappa \setminus X \text{ is bounded in } \kappa\} \subseteq U$.

Definition 1.2. Let F be a filter over a regular cardinal $\kappa \geq \omega$.

(1) F is λ*-complete* if F is closed under intersections of less than λ many of its members.
(2) F is *Ramsey* if for any function $f : [\kappa]^2 \to 2$ there is an $X \in F$ such that $f \restriction [X]^2$ is constant.
(3) F is *selective* if for every function $f : \kappa \to \kappa$, there is an $X \in F$ such that $f \restriction X$ is either constant or one-to-one.
(4) (Kanamori [21]) F is *rapid* if for each normal function $f : \kappa \to \kappa$ (i.e. increasing and continuous), there exists an $X \in F$ such that $\text{otp}(X \cap f(\alpha)) \leq \alpha$ for each $\alpha < \kappa$. (i.e. bounded pre-images), there is an $X \in F$ such that $|f^{-1}(\{\alpha\}) \cap X| \leq \alpha$ for every $\alpha < \kappa$.
(5) F is a *p-point* if whenever $f : \kappa \to \kappa$ is unbounded[10] on a set in F, it is almost one-to-one mod F, i.e. there is an $X \in F$ such that for every $\gamma < \kappa$, $|f^{-1}[\gamma] \cap X| < \kappa$.
(6) U is a *q-point* if every function $f : \kappa \to \kappa$ which is almost one-to-one mod F is injective mod F.

a κ*-filter* is a uniform, κ-complete filter.

The following facts are well known.

Fact 1.3. The following are equivalent for a κ-ultrafilter U:

(1) U is Ramsey.
(2) U is selective.
(3) U is a p-point and a q-point.

Fact 1.4. Suppose that U, V_α are ultrafilters on $\kappa \geq \omega$ where each V_α is uniform. Then $\sum_U V_\alpha$ is not a p-point.

[8]There are papers which consider the filter $\{A \times B \mid A \in F, B \in G\}$ and denote it by $F \times G$, this filter will not be considered in this paper so there is no risk of confusion.

[9]Or non-principal.

[10]Namely, $f^{-1}[\alpha] \notin F$ for every $\alpha < \kappa$

Indeed the function π_1, the projection to the first coordinate, is never almost one-to-one on a set in $X \in \sum_U V_\alpha$.

Definition 1.5. Let F, G be filters on X, Y resp. We say that F is *Rudin-Keisler* below G, denoted by $F \leq_{RK} G$, if there is a *Rudin-Keisler* projection $f : Y \to X$ such that

$$f_*(G) := \{A \subseteq X \mid f^{-1}[A] \in G\} = F$$

We say that are RK-isomorphic, and denote it by $F \equiv_{RK} G$ if there is a bijection f such that $f_*(F) = G$.

It is well known that if $F \leq_{RK} G \wedge G \leq_{RK} F$ then $F \equiv_{RK} G$ and that $F, G \leq_{RK} F \cdot G$ via the projection to the first and second coordinates respectively. Also, the Rudin-Keisler order implies the Tukey order. A Ramsey ultrafilter over κ is characterized as being Rudin-Keisler minimal among κ-ultrafilters.

Next, let us record some basic terminology and facts regarding cofinal types. Given two directed partially ordered sets $\mathbb{P}, \mathbb{Q}$, the Cartesian product $\mathbb{P} \times \mathbb{Q}$ ordered pointwise, is the least upper bound of $\mathbb{P}, \mathbb{Q}$ in the Tukey order (see [13]). It follows that $F \times G \leq_T F \cdot G$. More generally, for partially ordered sets $(\mathbb{P}_i, \leq_i)$ for $i \in I$, we denote by $\prod_{i \in I}(\mathbb{P}_i, \leq_i)$ to be the order over the underlining set $\prod_{i \in I} \mathbb{P}_i$ with the everywhere domination order, namely $f \leq g$ iff for all $i \in I$, $f(i) \leq_i g(i)$. If the order is clear from the context we omit it and just write $\prod_{i \in I} \mathbb{P}_i$. This is the case when $\mathbb{P}_i = U_i$ is a filter ordered by reversed inclusion of an ideal ordered by inclusion. If for every $i \in I$, $\mathbb{P}_i = \mathbb{P}$ we simply write $\mathbb{P}^I$.

1.2. **The cofinal type of sums and products.** The following theorem [14] provides the starting point for the analyses of the cofinal type of sums of ultrafilters:

Theorem 1.6 (Dobrinen-Todorcevic)**.** *Let F, G_x be filters as in Definition 1.1. Then:*

(1) $\sum_F G_x \leq_T F \times \prod_{x \in X} G_x$.
(2) If $G_x = G$ for every x, then $F \cdot G \leq_T F \times G^X$.
(3) If $F = G$, then $F \cdot F \leq_T F^X$.

These Tukey types are invariant under Rudin-Keisler isomorphic copies of the ultrafilters involved, hence we may assume for the rest of this paper that ultrafilters are defined on regular (infinite) cardinals. It was later discovered [29] that $(2), (3)$ of Theorem 1.6 are in fact an equivalences:

Theorem 1.7 (Milovich)**.** *Let F, G are κ-filters, then $F \cdot G \equiv_T F \times G^\kappa$ and in particular $F \cdot F \equiv_T F^\kappa$.*

The proof of Milovich's Theorem go through in case F is any ultrafilter over λ and G is λ-complete.

Corollary 1.8 (Milovich)**.** *For any two κ-filters F, G, $F \cdot (G \cdot G) \equiv_T F \cdot G$ and $F^3 \equiv_T F^2$.*

It is tempting to conjecture that $\sum_U V_\alpha \equiv_T U\times\prod_{\alpha<\lambda}V_\alpha$, however, this will not be the case in general, as indicated by the following example:

Example 1.9. Suppose that U and V are Tukey incomparable ultrafilters on ω, and $U\equiv_T U\cdot U$. This situation is obtained for example under $Cov(\mathcal{M})=\mathfrak{c}$ [11]. The incomparability requirement ensures that $U\times V>_T U$. Let $V_0=V$ and $V_n=U$ for $n>0$. Then

$$\sum_U V_n = U\cdot U\equiv_T U<_T U\times V\leq_T U\times\prod_{n<\omega}V_n.$$

The point of the example is that the sum is insensitive to removing a neglectable set of coordinates, while the product changes if we remove even a single coordinate. Another quite important difference is that $U\times\prod_{x\in X}V_x$ is insensitive to permutations of the indexing set, while $\sum_U V_x$ is. Formally, this is expressed by the following fact:

Fact 1.10. Let U be an ultrafilter over $\lambda\geq\omega$ and U_α on δ_α. For every $X\in U$, $\sum_U V_\alpha\leq_T U\times\prod_{\alpha\in X}V_\alpha\leq_T U\times\prod_{\alpha<\lambda}V_\alpha$.

Proof. The right inequality is clear. The left one is also simple, since the set $\mathcal{X}\subseteq\sum_U V_\alpha$, of all Y such that $\pi_1''Y\subseteq X$ is a cofinal set in $\sum_U V_\alpha$ and therefore the map $F:U\times\prod_{\alpha\in X}V_\alpha\to\sum_U V_\alpha$ defined by

$$F(\langle Z,\langle A_\alpha\mid\alpha\in X\rangle\rangle)=\bigcup_{\alpha\in Z\cap X}\{\alpha\}\times A_\alpha$$

is monotone and has cofinal image. □

In this section, we provide further insight into the cofinal type of $\sum_U V_\alpha$. We will focus on κ-ultrafilters, so our initial assumption is that U is a λ-ultrafilter for $\lambda\geq\omega$ and $\langle V_\alpha\mid\alpha<\lambda\rangle$ is a sequence of ultrafilters such that each V_α is a δ_α-ultrafilter where $\delta_\alpha\geq\omega$. Towards our first result, consider the set

$$\mathcal{B}(U,\langle V_\alpha\mid\alpha<\lambda\rangle)=\{U\times\prod_{\alpha\in X}V_\alpha\mid X\in U\}$$

ordered by the Tukey order. This is clearly a downward-directed set. Our goal is to prove that in some sense, $\sum_U V_\alpha$ is the greatest lower bound of $\mathcal{B}(U,\langle V_\alpha\mid\alpha<\lambda\rangle)$. Consider the maps

$$\pi_X:U\times\prod_{\alpha\in X}V_\alpha\to\sum_U V_\alpha,\qquad \pi_{X,Y}:U\times\prod_{\alpha\in X}V_\alpha\to U\times\prod_{\alpha\in Y}V_\alpha$$

Defined for $X,Y\in U$ where $Y\subseteq X$ defined by

$$\pi_X(\langle Z,\langle A_\alpha\mid\alpha\in X\rangle\rangle)=\bigcup_{\alpha\in X\cap Z}\{\alpha\}\times A_\alpha\ \text{ and}$$

$$\pi_{X,Y}(\langle Z,\langle A_\alpha\mid\alpha\in X\rangle\rangle)=\langle Z,\langle A_\alpha\mid\alpha\in Y\rangle\rangle.$$

[11]By Ketonen [22], this assumption implies that there are $(2^{\mathfrak{c}})^+$-many distinct selective ultrafilters. Then there are two Tukey incomparable selective ultrafilters and by Dobrinen and Todorcevic [14], $U\cdot U\equiv_T U$ for any selective ultrafilter.

Then

(1) π_X is monotone cofinal and $\text{rng}(\pi_X)$ is exactly all the sets $B \in \sum_U V_\alpha$ in standard form[12] such that $\pi'' B \subseteq X$.
(2) $\pi_{X,Y}$ is monotone cofinal.
(3) $\pi_Y \circ \pi_{X,Y}(C) \subseteq \pi_X(C)$.

Suppose that $\sum_U V_\alpha \geq_T \mathbb{P}$. Recall that if $\mathbb{P}$ is complete[13] (e.g. $\mathbb{P} = F$ is a filter ordered by reverse inclusion or any product of complete orders), then $\mathbb{Q} \geq_T \mathbb{P}$ implies that there is a monotone[14] cofinal map $f : \mathbb{Q} \to \mathbb{P}$. Suppose that $\mathbb{P}$ is complete and let $g : \sum_U V_\alpha \to \mathbb{P}$ be monotone cofinal. Define $f_X = g \circ \pi_X$. Then f_X is monotone cofinal from $U \times \prod_{\alpha \in X} V_\alpha$ to $\mathbb{P}$. Moreover, we have that if $Y \subseteq X$ then

$$f_Y(\pi_{X,Y}(C)) = g(\pi_Y(\pi_{X,Y}(C)) \geq_{\mathbb{P}} g(\pi_X(C)) = f_X(C)$$

Definition 1.11. A sequence of monotone cofinal maps

$$\langle f_X : U \times \prod_{\alpha \in X} V_\alpha \to \mathbb{P} \mid X \in U \rangle$$

if said to be *coherent* if

(†) whenever $Y \subseteq X$, and $C \in U \times \prod_{\alpha \in X} V_\alpha$, $f_Y(\pi_{X,Y}(C)) \geq_{\mathbb{P}} f_X(C)$.

A poset $\mathbb{P}$ is said to be *uniformly below* $\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda \rangle)$ if there is a coherent sequence of monotone cofinal maps $\langle f_X : U \times \prod_{\alpha \in X} V_\alpha \to \mathbb{P} \mid X \in U \rangle$.

The following theorem says that $\sum_U V_\alpha$ is the greatest lower bound among all the posts uniformly below $\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda \rangle)$.

Theorem 1.12. *Suppose that $\mathbb{P}$ is a complete order. Then $\mathbb{P}$ is uniformly below $\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda \rangle)$ if and only if $\sum_U V_\alpha \geq_T \mathbb{P}$.*

Proof. From right to left was already proven in the paragraph before Definition 1.11. Let us prove from left to right. Let $\langle f_X \mid X \in U \rangle$ be the sequence witnessing that $\mathbb{P}$ is uniformly below $\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda \rangle)$. Let $\mathcal{X} \subseteq \sum_U V_\alpha$ be the usual cofinal set of all the sets $A \in \sum_U V_\alpha$ is a standard form. Let us define $F : \chi \to \mathbb{P}$ monotone and cofinal,

$$F(A) = f_{\pi_1'' A}(\langle \pi_1'' A, \langle (A)_\alpha \mid \alpha \in \pi'' A \rangle \rangle)$$

We claim first (and most importantly) that F is monotone. Suppose that $A, B \in \chi$ are such that $A \subseteq B$. Then,

(a.) $\pi_1'' A \subseteq \pi_1'' B$ and

[12] A set $B \in \sum_U V_\alpha$ is said to be in *standard form* if for every $\alpha < \lambda$, either $(B)_\alpha = \emptyset$ or $(B)_\alpha \in V_\alpha$.

[13] i.e., every bounded subset of $\mathbb{P}$ has a least upper bound.

[14] $f : \mathbb{Q} \to \mathbb{P}$ is called monotone if $q_1 \leq_{\mathbb{Q}} q_2 \Rightarrow f(q_1) \leq_{\mathbb{P}} f(q_2)$.

(b.) for every $\alpha<\lambda$, $(A)_\alpha\subseteq(B)_\alpha$.

Define the sequence $\langle X_\alpha\mid\alpha\in\pi_1''B\rangle$ by $X_\alpha=(A)_\alpha$ for $\alpha\in\pi_1''A$ and $X_\alpha=(B)_\alpha$ for $\alpha\in\pi_1''B\setminus\pi_1''A$. Note that

$$\pi_{\pi_1''B,\pi_1''A}(\langle\pi_1''A,\langle X_\alpha\mid\alpha\in\pi_1''B\rangle\rangle)=\langle\pi_1''A,\langle(A)_\alpha\mid\alpha\in\pi_1''A\rangle\rangle$$

and that $X_\alpha\subseteq(B)_\alpha$. It follows by monotonicity of the functions, and by (†) that

$$F(A)=f_{\pi_1''A}(\langle\pi_1''A,\langle(A)_\alpha\mid\alpha\in\pi_1''A\rangle\rangle)=f_{\pi_1''A}(\pi_{\pi_1''B,\pi_1''A}(\langle\pi_1''A,\langle X_\alpha\mid\alpha\in\pi_1''B\rangle\rangle)$$

$$\geq_{\mathbb{P}} f_{\pi_1''B}(\langle\pi_1''A,\langle X_\alpha\mid\alpha\in\pi_1''B\rangle\rangle)\geq_{\mathbb{P}} f_{\pi_1''B}(\langle\pi_1''B,\langle(B)_\alpha\mid\alpha\in\pi_1''B\rangle\rangle=F(B)$$

To see it is cofinal, let $p\in\mathbb{P}$ be any element, fix any $X\in U$, since f_X is cofinal, there is $\langle Z,\langle A_\alpha\mid\alpha\in X\rangle\rangle\in U\times\prod_{\alpha\in X}V_\alpha$ such that $f_X(\langle Z,\langle A_\alpha\mid\alpha\in X\rangle\rangle)\geq_{\mathbb{P}} p$. Consider $A=\cup_{\alpha\in Z\cap X}\{\alpha\}\times A_\alpha$. Then

$$F(A)=f_{Z\cap X}(\langle Z\cap X,\langle A_\alpha\mid\alpha\in Z\cap X\rangle\rangle)=$$

$$f_{Z\cap X}(\pi_{X,Z'\cap X}(\langle\langle Z\cap X,\langle A_\alpha\mid\alpha\in X\rangle\rangle)))\geq_{\mathbb{P}} f_X(\langle Z\cap X,\langle A_\alpha\mid\alpha\in X\rangle)\geq_{\mathbb{P}} p$$

□

Lemma 1.13. *Suppose that $\mathbb{P}$ is complete and for each $X\in U$, $\mathcal{X}_X\subseteq U\times\prod_{\alpha\in X}V_\alpha$ is such that:*

(1) $\mathcal{X}_X$ is a cofinal subset of $U\times\prod_{\alpha\in X}V_\alpha$.
(2) $f_X:\mathcal{X}_X\to\mathbb{P}$ is monotone cofinal.
(3) whenever $Y\subseteq X$, $\pi''_{X,Y}\mathcal{X}_X\subseteq\mathcal{X}_Y$ and $f_Y(\pi_{X,Y}(C))\geq_{\mathbb{P}} f_X(C)$

Then $\mathbb{P}$ is uniformly below $\mathcal{B}(U,\langle B_\alpha\mid\alpha<\lambda\rangle)$.

Proof. Define $f^*_X:U\times\prod_{\alpha\in X}V_\alpha\to\mathbb{P}$ by

$$f^*_X(A)=\sup\{f_X(B)\mid A\subseteq B\in\mathcal{X}_X\}.$$

Note that if $B'\in\mathcal{X}_X$ is such that $B'\subseteq A$, then the set $\{f_X(B)\mid A\subseteq B\in\mathcal{X}_X\}$ is bound in $\mathbb{P}$ by $f_X(B')$ (as f_X is monotone). Hence $f^*_X(A)$ is well defined by completeness of $\mathbb{P}$. It is straightforward that Since f_X is monotone cofinal, f^*_X is monotone cofinal. To see (†), suppose that $Y\subseteq X$, and $C\in U\times\prod_{\alpha<\lambda}V_\alpha$, then for every $C\subseteq B\in\mathcal{X}_X$, then by (3) $\pi_{X,Y}(C)\subseteq\pi_{X,Y}(B)\in\mathcal{X}_X$ and $f_Y(\pi_{X,Y}(B))\geq_{\mathbb{P}} f_X(B)$. If follows that

$$f^*_X(C)=\sup\{f_X(B)\mid C\subseteq B\in\mathcal{X}_X\}\leq\sup\{f_Y(\pi_{X,Y}(B))\mid C\leq_{\mathbb{P}} B\in\mathcal{X}_X\}$$

$$\leq_{\mathbb{P}}\sup\{f_Y(B')\mid\pi_{X,Y}(C)\subseteq B'\in\mathcal{X}_Y\}=f^*_Y(\pi_{X,Y}(C)).$$

Hence $\langle f^*_X\mid X\in U\rangle$ is coherent and therefore $\mathbb{P}$ is uniformly below $\mathcal{B}(U,\langle V_\alpha\mid\alpha<\lambda\rangle)$. □

Corollary 1.14. *Let U be an ultrafilter on $\lambda\geq\omega$ and that each V_α is a δ_α-complete ultrafilter on some $\delta_\alpha>\alpha$. $\mathbb{P}\leq_T V_\alpha$ for every $\alpha<\lambda$, then $\mathbb{P}^\lambda\leq_T\sum_U V_\alpha$.*

Proof. We fix for every $\alpha < \lambda$, $f_\alpha : V_\alpha \to \mathbb{P}$ monotone and cofinal. Now for every $X \in U$, we define a cofinal set $\mathcal{X}_X \subseteq U \times \prod_{\alpha \in X} V_\alpha$ consisting of all the elements $\langle Z, \langle A_\alpha \mid \alpha \in X\rangle\rangle \in U \times \prod_{\alpha\in X} V_\alpha$ such that for every $\alpha < \beta$ in X, $f_\alpha(A_\alpha) \leq_{\mathbb{P}} f_\beta(A_\beta)$.

Claim 1.15. *$\mathcal{X}_X$ is cofinal in $U \times \prod_{\alpha\in X} V_\alpha$*

Proof. Let $\langle Z, \langle B_\alpha \mid \alpha \in X\rangle\rangle$, let us construct A_α recursively. Let $B_0 = A_0$. Suppose that A_α for $\alpha \in X \cap \beta$ where defined for some $\beta \in X$. Then for each $\alpha \in X \cap \beta$, we find (by cofinality of f_β) a set $C_\alpha \in V_\beta$ such that $f_\alpha(A_\alpha) \leq_{\mathbb{P}} f_\beta(C_\alpha)$. Let $A_\beta = B_\beta \cap \bigcap_{\alpha<\beta} C_\alpha$. By δ_β-completeness of V_β, $A_\beta \in V_\beta$. By monotonicity of f_β, we conclude that for every $\alpha < \beta$, $f_\alpha(A_\alpha) \leq_{\mathcal{P}} f_\beta(A_\beta)$. It is now clear that $\langle Z, \langle A_\alpha \mid \alpha \in X\rangle\rangle \in \mathcal{X}_X$ and above $\langle Z, \langle B_\alpha \mid \alpha \in X\rangle\rangle$. □

Note that $\pi''_{X,Y}\mathcal{X}_X \subseteq \mathcal{X}_Y$. Define $f_X : \mathcal{X}_X \to \mathbb{P}^\lambda$ by

$$f_X(\langle Z, \langle A_\alpha \mid \alpha \in X\rangle\rangle) = \langle f_{X(\alpha)}(A_{X(\alpha)}) \mid \alpha < \lambda\rangle.$$

Let $\langle Z, \langle A_\alpha \mid \alpha \in X\rangle\rangle \in \mathcal{X}_X$, and let $Y \subseteq X$, then $Y(\alpha) \geq X(\alpha)$. Hence, by definition of $\mathcal{X}_X$, $f_{X(\alpha)}(A_{X(\alpha)}) \leq_{\mathbb{P}} f_{Y(\alpha)}(A_{Y(\alpha)})$. We conclude that

$$\begin{aligned} f_X(\langle Z, \langle A_\alpha \mid \alpha \in X\rangle\rangle) &= \langle f_{X(\alpha)}(A_{X(\alpha)}) \mid \alpha < \lambda\rangle \leq_{\mathbb{P}^\omega} \langle f_{Y(\alpha)}(A_{Y(\alpha)}) \mid \alpha < \lambda\rangle \\ &= f_Y(\langle Z, \langle A_\alpha \mid \alpha \in Y\rangle\rangle) = f_Y(\pi_{X,Y}(\langle Z, \langle A_\alpha \mid \alpha \in X\rangle\rangle)) \end{aligned}$$

Hence by Lemma 1.13 $\mathbb{P}^\lambda$ is uniformly below $\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda\rangle)$ and by Theorem 1.12, $\mathbb{P}^\lambda \leq_T \sum_U V_\alpha$. □

In particular, U_α is Tukey-top for a set of α's in U, then $\sum_U U_\alpha$ is Tukey top.

It is unclear whether being uniformly below $\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda\rangle)$ is equivalent to simply being a Tukey below each $X \in \mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda\rangle)$. Hence it is unclear if $\sum_U V_\alpha$ is indeed the greatest lower bound of $\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda\rangle)$ in the usual sense; if every $\mathbb{P}$ which is a lower bound in the Tukey order for $\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda\rangle)$ is Tukey below $\sum_U V_\alpha$. Let us give a few common configurations of the Tukey relation among the ultrafilters V_α in which $\sum_U V_\alpha$ is the greatest lower bound in the usual sense. Let us denote that by $\sum_U V_\alpha = \inf(\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda\rangle))$.

The following is a straightforward corollary from Theorem 1.12:

Corollary 1.16. *Let $X_0 \in U$, then $\sum_U V_\alpha \equiv_T U \times \prod_{\alpha\in X_0} V_\alpha$ if and only if $U \times \prod_{\alpha\in X_0} V_\alpha$ is uniformly below $\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda\rangle)$. In that case $\sum_U V_\alpha = \inf(\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda\rangle)$.*

The second case in which $\sum_U V_\alpha$ turns out to be the greatest lower bound is the following:

Lemma 1.17. *Suppose that there is a set $X_0 \in U$ such that for every $\alpha < \beta \in X_0$, V_α is a κ-complete ultrafilter such that $V_\alpha \cdot V_\alpha \equiv_T V_\alpha >_T V_\beta$. Then $\sum_U V_\alpha = \inf(\mathcal{B}(U, \langle V_\alpha \mid \alpha < \lambda\rangle))$ is a strict greatest lower bound.*

Proof. First note that for every $Y\subseteq X$, by the assumptions,

$$V_{\min(Y)}\leq_T\prod_{m\in Y}V_m\leq_T\prod_{m\in Y}V_{\min(Y)}\equiv_T V_{\min(Y)}\cdot V_{\min(Y)}\equiv_T V_{\min(Y)}.$$

Therefore, if $\mathbb{P}\leq_T B$ for every $B\in\mathcal{B}(U,\langle V_\alpha\mid\alpha<\lambda\rangle)$ then $\mathbb{P}\leq_T V_\alpha$ for every $\alpha\in X$. By corollary 1.14, it follows than that $\mathbb{P}\leq_T\sum_U V_\alpha$. Moreover, $\sum_U V_\alpha$ is strictly below $\mathcal{B}(U,\langle V_\alpha\mid\alpha<\lambda\rangle)$, since for every $\beta<\lambda$, $\sum_U V_\alpha\leq_T V_{\beta+1}<_T V_\beta$. □

We will later show that the assumptions in the above Lemma are consistent. Before that, we consider the third configuration in which the ultrafilters are increasing, the proof below works only for ultrafilters on ω and we do not know whether it is possible to generalize it to other ultrafilters.

Lemma 1.18. *Suppose that U,V_n are ultrafilters on ω, such that on a set $X_0\in U$, for every $n\leq m\in X_0$, $V_n\leq_T V_m$. Then*

$$U\times\prod_{n\in X_0}V_n\equiv_T\sum_U V_n=\inf(\mathcal{B}(U,\langle V_\alpha\mid\alpha<\lambda\rangle))$$

Proof. By Corollary 1.16, if $\sum_U V_n\equiv_T U\times\prod_{n\in X_0}V_\alpha$, then it must be the greatest lower bound as well. To prove the Tukey-equivalence, first note that $\sum_U V_n\leq_T U\times\prod_{n\in X_0}V_n$ by Fact 1.10. For the other direction, define for every $n\in X_0$, $n^+=\min(X_0\setminus n+1)$ and let $f_{n^+,n}:V_{n^+}\to V_n$ monotone and cofinal. Denote by $n^{+2}=(n^+)^+$ and $n^{+k}=(n^{+(k-1)})^+$ be the k^{the} successor of n in X_0. For any $n<m\in X_0$, suppose that $m=n^{+k}$ and let $f_{m,n}=f_{n^{+k},n^{+(k-1)}}\circ f_{n^{+(k-1)},n^{+(k-1)}}\circ ...\circ f_{n^+,n}$. Moreover, let $f_{n,n}:V_n\to V_n$ be the identity. Hence $f_{m,n}:V_m\to V_n$ is monotone cofinal, and if $k\in X_0\cap[n,m]$ then $f_{m,n}=f_{k,n}\circ f_{m,k}$.

Let us define a coherent sequence of cofinal maps from a cofinal subset of $U\times\prod_{n\in X}V_n$ to $U\times\prod_{n\in X_0}V_n$ for $X\in U$. Consider the collection $\mathcal{X}_X\subseteq U\times\prod_{n\in X}V_n$ of all $\langle Z,\langle A_n\mid n\in X\rangle\rangle$ such that for all $n,m\in X\cap X_0$, if $n<m$ then $f_{m,n}((A)_m)\subseteq(A)_n$. It is straightforward to check that if $Y\subseteq$ then $\pi''_{X,Y}\mathcal{X}_X\subseteq\mathcal{X}_Y$.

Claim 1.19. *$\mathcal{X}_X$ is cofinal in $U\times\prod_{n\in X}V_n$.*

Proof. Let $\langle Z,\langle A_n\mid n\in X\rangle\rangle\in U\times\prod_{n\in X}V_n$. We define a sequence X_n be induction on $n\in X$. $X_{n_0}=(A)_{n_0}$. Suppose we have defined $X_{n_k}\in V_{n_k}$ for some $k<m$, For each k, we find $C_{m,k}\in V_m$ such that $f_{m,k}(C_{m,k})\subseteq X_{n_k}$. Define $X_{n_m}=(A)_{n_m}\cap(\bigcap_{k<m}C_{m,k})$. By monotonicity of the $f_{m,k}$'s $f_{m,k}(X_{n_m})\subseteq X_{n_k}$. Let $A_1=\langle Z,\langle X_n\mid n\in X\rangle\rangle$, then $A_1\in\mathcal{X}$ and $A_1\geq\langle Z,\langle A_\alpha\mid n\in X\rangle\rangle$. □

Fix the unique order isomorphism $\sigma_{X,X_0}:X\to X_0$ (which then satisfy $\sigma(n)\leq n$ as $X\subseteq X_0$) and let $f_X:\mathcal{X}_X\to U\times\prod_{n\in X_0}V_n$ be defined by

$$f_X(\langle Z,\langle A_n\mid n\in X\rangle\rangle)=\langle Z,\langle f_{\sigma^{-1}_{X,X_0}(n),n}((A)_{\sigma^{-1}_{X,X_0}(n)})\mid n\in X_0\rangle\rangle.$$

Clearly, f_X is monotone, let us check that it is cofinal and that the sequence $\langle f_X \mid X \in U \restriction X_0\rangle$ is coherent. Suppose that $C_1 = \langle Z, \langle A_n \mid n \in X_0\rangle\rangle \in U \times \prod_{n\in X_0} V_n$. , We find $\langle B_n \mid n \in X_0\rangle \geq \langle A_n \mid n \in X_0\rangle$ such that $f_{m,n}(B_m) \subseteq B_n$ for every $n < m \in X_0$. This is possible as before, constructing the B_n's by induction and the fact that at each step we only have finitely many requirements, so we can intersect the corresponding finitely many sets. Now take $\langle Z, \langle B_n \mid n \in X\rangle\rangle \in \mathcal{X}_X$. Then

$$f_X(\langle Z, \langle B_n \mid n \in X\rangle\rangle) = \langle Z, \langle f_{\sigma^{-1}_{X,X_0}(n),n}(B_{\sigma^{-1}_{X,X_0}(n)}) \mid n \in X_0\rangle\rangle.$$

Since $\sigma^{-1}_{X,X_0}(n) \geq n$ for every $n \in X_0$, we conclude that

$$f_{\sigma^{-1}_{X,X_0}(n),n}(B_{\sigma^{-1}_{X,X_0}(n)}) \subseteq B_n,$$

and therefore f_X is cofinal. Similarly, to see (†), if $Y \subseteq X \subseteq X_0$, and $\langle Z, \langle A_n \mid n \in X\rangle\rangle \in \mathcal{X}_X$, then $\sigma^{-1}_{Y,X_0}(n) \geq \sigma^{-1}_{X,X_0}(n)$, and therefore, for every $n < \omega$,

$$f_{\sigma^{-1}_{Y,X_0}(n),n}((A)_{\sigma^{-1}_{Y,X_0}(n)}) = f_{\sigma^{-1}_{X,X_0}(n),n}(f_{\sigma^{-1}_{Y,X_0}(n),\sigma^{-1}_{X,X_0}(n)}(A_{\sigma^{-1}_{Y,X_0}(n)})) \subseteq$$
$$\subseteq f_{\sigma^{-1}_{X,X_0}(n),n}(A_{\sigma^{-1}_{X,X_0}(n)}).$$

It follows that $f_Y(\pi_{X,Y}(\langle Z, \langle A_n \mid n \in X\rangle\rangle)) \geq f_X(\langle Z, \langle A_n \mid n \in X\rangle\rangle)$ □

The above Lemma recovers Milovich's theorem 1.7, taking each $V_n = V$ for every n.

Our next goal is to prove that the assumptions of Lemma 1.17 are consistent. This example shows that the cofinal type of $\sum_U V_n$ in general can be quite complicated. To do that, we will need a theorem of Raghavan and Todorcevic from [31] regarding the canonization of cofinal maps from basically generated ultrafilters. The notion of basically generated ultrafilters was introduced by Dobrinen and Todorcevic [14] as an attempt to approximate the class of ultrafilters which are not Tukey-top. Recall that an ultrafilter U is called *basically generated* if there is a cofinal set $\mathcal{B} \subseteq U$ such that for every sequence $\langle b_n \mid n < \omega\rangle \subseteq \mathcal{B}$ which converges[15] to an element of $\mathcal{B}$, there is $I \in [\omega]^\omega$ such that $\cap_{i\in I} A_i \in U$. A p-point ultrafilter U is basically generated as witnessed by $\mathcal{B} = U$ ([14, Thm. 14]). Dobrinen and Todorcevic proved that produces and sums of p-points must also be basically generated ([14, Thm. 16]).

Theorem 1.20 (Raghavan-Todorcevic)**.** *Let U be a basically generated ultrafilter and V be any ultrafilter such that $V \leq_T U$. Then there is $P \subseteq \mathrm{FIN}$ such that:*

(1) $\forall t, s \in P,\ t \subseteq s \Rightarrow t = s$.
(2) V is Rudin-Keisler below $U(P)$, namely, there is $f : P \to \omega$ such that $V = \{X \subseteq \omega \mid f^{-1}[X] \in U(P)\}$.

[15]A sequence $\langle A_n \mid n < \omega\rangle$ of subsets ω is said to converge to A if for every $n < \omega$ there is $N < \omega$ such that for every $m \geq N$, $A_m \cap n = A \cap n$.

(3) $U(P) \equiv_T V$.

Where is the filter $U(P) = \{A \subseteq P \mid \exists a \in U.[a]^{<\omega} \subseteq A\}$.

The forcing notion $P(\omega)/\text{fin}$ consists of infinite sets, ordered by inclusion up to a finite set. Namely, $X \leq^* Y$ if $X \setminus Y$ is finite. In the next proposition, we consider the forcing notion $\mathbb{P} = \prod_{n<\omega} P(\omega)/\text{fin}$, where elements of the product have full support. For more information regarding forcing we refer the reader to [23].

The following items summarize the properties of $\mathbb{P}$ which we will need:

- $\mathbb{P}$ is σ-closed, and therefore does not add new subsets of ω, and ω_1 is preserved.
- For each n, the projection π_n of $\mathbb{P}$ to the n^{th} coordinate is a forcing projection from $\mathbb{P}$ to $P(\omega)/\text{fin}$[16].
- If $G \subseteq \mathbb{P}$ is generic over V, then $U_n := \overline{\pi_n'' G} = \{X \in P(\omega) \mid \exists f \in G.f(n) \leq^* X\}$ is an ultrafilter over ω in $V[G]$. Moreover, U_n is a generic ultrafilter for $P(\omega)/\text{fin}$.
- Each U_n is a selective ultrafilter
- $U_n \notin V[\langle U_m \mid m \in \omega \setminus \{n\}\rangle]$.

Proposition 1.21. *Let* $\mathbb{P}$ *be a full support product of* ω*-copies of* $P(\omega)/\text{fin}$*. Let* $G \subseteq \mathbb{P}$ *be generic over* V*. Then in* $V[G]$ *there is a sequence of ultrafilters* V_n*, such that* $V_0 >_T V_1 >_T V_2 ...$ *and* $V_n \cdot V_n \equiv V_n$*.*

Proof. For each $n < \omega$, U_n is a selective ultrafilter and therefore by Theorem 0.4, $U_n \cdot U_n \equiv_T U_n \equiv_T (U_n)^\omega$. For every $n < \omega$, define[17]

$$V_n = \sum_{U_0} (U_{n+1} \cdot U_{n+2} \cdot ... \cdot U_{n+m})_{0<m<\omega}$$

Note that each $U_{n+1} \cdot ... \cdot U_{n+m}$ is basically generated as the product of p-points. Therefore, V_n is also basically generated.

Lemma 1.22. *(1)* $V_n \equiv_T U_0 \times \prod_{n<m<\omega} U_m$.

(2) $V_n \cdot V_n \equiv_T V_n$.

(3) $V_0 >_T V_1 >_T V_2 ...$.

Proof of Lemma. For (1), we note that the ultrafilters $U_{n+1} \cdot ... \cdot U_{n+m}$ over which we sum in the definition of V_n are increasing in the Tukey order. Hence by Lemma 1.18

$$V_n \equiv_T U_0 \times \prod_{0<m<\omega} U_{n+1} \cdot ... \cdot U_{n+m}$$

By Milovich's Theorem 1.7, and by our assumptions, for each n, m

$$U_{n+1} \cdot ... \cdot U_{n+m} \equiv_T U_{n+1} \times U_{n+2} \cdot U_{n+2} \times ... \times U_{n+m} \cdot U_{n+m} \equiv_T U_{n+1} \times ... \times U_{n+m}.$$

[16]A function from $f : \mathbb{P} \to \mathbb{Q}$ is called a projection of forcing notions if f is order-preserving, $\text{rng}(f)$ is dense in $\mathbb{Q}$, and for every $p \in \mathbb{P}$ and $q \leq_{\mathbb{Q}} p$, there is $p' \leq_{\mathbb{P}} p$ such that $f(p') \leq_{\mathbb{Q}} q$.

[17]We thank Gabe Goldberg for pointing out this definition of V_n.

Hence

$$V_n \equiv_T U_0 \times \prod_{0<m<\omega} U_{n+1} \times ... \times U_{n+m} \equiv_T U_0 \times \prod_{0<m<\omega} (U_{n+m})^\omega \equiv_T$$

$$\equiv_T U_0 \times \prod_{0<m<\omega} U_{n+m} \cdot U_{n+m} \equiv_T U_0 \times \prod_{0<m<\omega} U_{n+m}$$

Now for (2), we use (1). For each $n < \omega$

$$V_n \cdot V_n \equiv_T (V_n)^\omega \equiv_T (U_0 \times \prod_{0<m<\omega} U_{n+m})^\omega \equiv_T (U_0)^\omega \times \prod_{0<m<\omega} (U_{n+m})^\omega \equiv_T$$

$$\equiv_T U_0 \times \prod_{0<m<\omega} U_{n+m} \equiv_T V_n$$

For (3), it follows from (1) that

$$V_0 \geq_T V_1 \geq_T V_2 ....$$

Suppose toward a contradiction that $V_n \equiv_T V_{n+1}$ for some n. Then $U_{n+1} \leq_T V_{n+1}$. Note that

$$V_{n+1} \in V[U_0, \langle U_m \mid n+1 < m < \omega\rangle].$$

By mutual genericity $U_{n+1} \notin V[U_0, \langle U_m \mid n+1 < m < \omega\rangle]$. Since V_{n+1} is basically generated, Theorem 1.20 implies that there is $P \subseteq FIN$ such that $U_{n+1} \leq_{RK} V_n(P)$. Note that since $\mathbb{P}$ is σ-closed, $P \in V$ and therefore $V_n(P) \in V[U_0, \langle U_m \mid n+1 < m < \omega\rangle]$. Also the Rudin-Keisler projection f such that $f_*(V_n(P)) = U_{n+1}$ is in the ground model and therefore $U_{n+1} \in V[U_0, \langle U_m \mid n+1 < m < \omega\rangle]$, contradiction. □

□

It follows that $\sum_{U_0} V_n = \inf(\mathcal{B}(U_0, \langle V_n \mid 0 < n < \omega\rangle))$ is a strict greatest lower bound. Let us also prove that $U_0 <_T \sum_U V_n$. We will need the following folklore fact.

Fact 1.23. Suppose that $\sum_U V_n = \sum_U V'_n$ then $\{n < \omega \mid V_n = V'_n\} \in U$

Proof. Just otherwise, $Y = \{n < \omega \mid V_n \neq V'_n\} \in U$, in which case, for every $n \in Y$ take $X_n \in V_n$ such that $X_n^c \in V'_n$. Then $A = \bigcup_{n \in Y} \{n\} \times X_n \in \sum_U V_n$, while $A' = \bigcup_{n \in Y} \{n\} \times X_n^c \in \sum_U V'_n$. However $A \cap A' = \emptyset$ which contradicts $\sum_U V_n = \sum_U V'_n$. □

Proposition 1.24. $U_0 <_T \sum_{U_0} V_n$

Proof. Otherwise, there would have been a continuous cofinal map $f : U_0 \to \sum_{U_0} V_n$. Since U_0 is a selective ultrafilter, by Todorcevic [31], if $V \leq_T U_0$, then there is $\alpha < \omega_1$ such that $V =_{RK} U_0^\alpha$ for some $\alpha < \omega_1$. It follows that $\sum_{U_0} V_n \equiv_{RK} U_0^\alpha$ for some $\alpha < \omega_1$. If $\alpha > 1$, then $U_0^\alpha = \sum_{U_0} U_0^{\alpha_n}$ for some $\alpha_n < \alpha$ (The α_n's might be constant). It follows that $Y = \{n < \omega \mid V_n =_{RK} U_0^{\alpha_n}\} \in U_0$. Since for any $\beta < \omega_1$, $U_0^\beta \in V[U_0]$, for any $0 < n \in Y$, we conclude that $V_n \in V[U_0]$ and in particular $U_1 \in V[U_0]$, contradicting the mutual genericity. If $\alpha = 1$, then $U_0 =_{RK} \sum_{U_0} V_n$ which then implies that $\sum_{U_0} V_n$ is a p-point, in contradiction to Fact 1.4.

□

2. Two properties of filter

In this section, we present two properties of filters which play a key role in the proof of our main result. The first is the I-p.i.p which was introduced in [3], and the second is a new concept called *deterministic* ideals. Both of them provide an abstract framework in which one can analyze the connection between the Tukey type of an ultrafilter and ideals related to it. We start this section with a systematic study of the I-p.i.p. Many of our results in this section generalize to κ-filters for $\kappa \geq \omega$. However, we will restrict our attention to ultrafilters on ω, as the main application is the main result regarding the commutativity of Fubini products which is already known for κ-complete ultrafilters on measurable cardinals [2].

2.1. **The pseudo intersection property relative to a set.** Given set $\mathcal{F} \subseteq P(X)$, we denote by $\mathcal{F}^* = \{X \setminus A \mid A \in \mathcal{F}\}$. When $\mathcal{F}$ is a filter, $\mathcal{F}^*$ is an ideal which we call *the dual ideal*, and when $\mathcal{I}$ is an ideal $\mathcal{I}^*$ is a filter which we call *the dual filter*. Ideals are always considered with the (regular) inclusion order.

Fact 2.1. For every filter F, $(F, \subseteq) \simeq (F^*, \supseteq)$ and in particular $(F, \subseteq) \equiv_T (F^*, \supseteq)$.

Definition 2.2. A filter F over a countable set S such that $T \subseteq F^*$ (T is any subset), is said to satisfy the T-pseudo intersection property (T-p.i.p) if for every sequence $\langle X_n \mid n < \omega \rangle \subseteq F$, there is $X \in F$ such that for every n, there is $t \in T$ such that $X \setminus X_n \subseteq t$.

The proof for these simple facts can be found in [3]:

Fact 2.3.
(1) Any filter F has the F^*-p.i.p.
(2) F is a p-point iff F has fin-p.i.p

The following facts are also easy to verify:

Fact 2.4.
(1) If T is downward closed with respect to inclusion, then F has the T-p.i.p if and only if for every sequence $\langle X_n \mid n < \omega \rangle \subseteq F$, there is $X \in F$ such that $X \setminus X_n \in T$.
(2) F has $\{\emptyset\}$-p.i.p if and only if F is σ-complete (and therefore, if F is on ω, then it is principal).

Benhamou and Dobrinen proved the following:

Proposition 2.5 ([3])**.** *Suppose that F is a filter and $I \subseteq F^*$ is any ideal such that F has the I-p.i.p. Then $F^\omega \leq_T F \times I^\omega$.*

Theorem 2.6 ([3])**.** *Suppose that U is an ultrafilter and $I \subseteq U^*$ is an ideal such that:*

(1) U has the I-p.i.p.
(2) $I^\omega \leq_T U$
Then $U \cdot U \equiv_T U$.

This theorem is the important direction in the equivalence of Theorem 0.5. This subsection is devoted to a systematic study of this property, which will be used in the proof for our main theorem regarding the commutativity of the cofinal types of Fubini products of ultrafilters. First let us provide an equivalent condition to being I-p.i.p, similar to the one we have for p-points.

Proposition 2.7. *Let U be any ultrafitler. Then the following are equivalent:*

(1) U has the I-p.i.p
(2) For any partition $\langle A_n \mid n < \omega\rangle$ such that for any n, $A_n \notin U$, there is $A \in U$ such that $A \cap A_n \in I$ for every $n < \omega$.
(3) Every function $f : \omega \to \omega$ which is unbounded modulo U is I-to-one modulo U, i.e. there is $A \in U$ such that for every $n < \omega$, $f^{-1}[n + 1] \cap A \in I$.

Proof. The proof is standard and is just a generalization of the usual characterization of p-points.

(1) ⇒ (2) Let $\langle A_n \mid n < \omega\rangle$ be a partition such that $A_n \notin U$. Let $B_n = \omega \setminus A_n \in U$ and by the I-p.i.p there is $A \in U$ such that $A \setminus B_n \in I$. It remains to note that $A \setminus B_n = A \cap A_n$ to conclude (2).

(2) ⇒ (3) Let $f : \omega \to \omega$ be unbounded modulo U. Let $A_n = f^{-1}[\{n\}]$, then $A_n \notin U$. Apply (2) to the partition $\langle A_n \mid n < \omega\rangle$ to find $A \in U$ such that $A \cap A_n \in I$. For any $n < \omega$, $f^{-1}[n+1] \cap A = \cup_{m \leq n} f^{-1}[\{m\}] \cap A \in I$. Hence f is I-to-one modulo U.

(3) ⇒ (1) Let $\langle B_n \mid n < \omega\rangle \subseteq U$, and let us assume without loss of generality that it is $\subseteq$-decreasing and that $\bigcap_{n<\omega} B_n = \emptyset$. Define $f(n) = \min\{m \mid n \notin B_m\}$. Since $\bigcap_{n<\omega} B_n = \emptyset$, $f : \omega \to \omega$ is a well defined function. Apply (3), to find $A \in U$ such that for every $n < \omega$ $f^{-1}[n+1] \cap A \in I$. Now for each $x \in A \setminus B_n$, $f(x) \leq n$ and therefore $x \in f^{-1}[n + 1] \cap A$ and therefore $A \setminus B_n \subseteq f^{-1}[n + 1] \cap A \in I$. It follows that $A \setminus B_n \in I$ and that U has the I-p.i.p.

□

Proposition 2.8. *(1) If F has T-p.i.p and $T \subseteq S$, then F has S-p.i.p.*
(2) Suppose that $T_1, T_2 \subseteq F^*$ are downwards-closed with respect to inclusion, and F has both the T_1-p.i.p and the T_2-p.i.p, then F has the $T_1 \cap T_2$-p.i.p
(3) Suppose that $f_*(G) = F$ and G has the T-p.i.p then F has the $\{f''t \mid t \in T\}$-p.i.p.

Proof. For (1), see [3]. For (2), suppose that U has both the T_1-p.i.p and the T_2-p.i.p. and T_1, T_2 are downwards closed. Let $\langle A_n \mid n < \omega\rangle$ be a sequence, then there are $A, B \in F$ such that for every n, $A \setminus A_n \in T_1$ and $B \setminus A_n \in T_2$. It follows that $A \cap B \in F$, fix $n < \omega$, then $A \cap B \setminus A_n$ is included in both $A \setminus A_n$ and $B \setminus A_n$ which implies that $A \cap B \setminus A_n \in T_1 \cap T_1$ as both T_1, T_2 are downwards closed. Hence F has the $T_1 \cap T_2$-p.i.p.

For (3), let $\langle X_n \mid n < \omega \rangle \subseteq F$, then $\langle f^{-1}[X_n] \mid n < \omega \rangle \subseteq G$. Therefore, there is $Y \in G$ such that for every n there is $t_n \in T$ such that $Y \setminus f^{-1}[X_n] \subseteq t_n$. Let $X = f''Y \in F$, we have that

$$X \setminus X_n \subseteq f''[Y \setminus f^{-1}[X_n]] \subseteq f''t_n$$

□

Corollary 2.9. *Let F be any filter. Denoted by $U_F \subseteq P(F^*)$ the set generated by all T's such that F has the T-p.i.p, namely,*

$$U_F = \{S \in P(F^*) \mid \exists T \subseteq S \text{ downwards closed } F \text{ has } T\text{-p.i.p}\}.$$

Then U_F is a filter over F^*.

U_F is almost an ultrafilter:

Proposition 2.10. *Suppose that $X_1, X_2 \subseteq P(F^*)$, and F has the $X_1 \cup X_2$-p.i.p, then either F has the X_1-p.i.p or X_2-p.i.p*

Proof. Suppose otherwise that X_1, X_2 are downward closed, F has the $X_1 \cup X_2$-p.i.p , but does not have the neither the X_1-p.i.p nor the X_2-p.i.p. Then there are sequences $\langle A_n \mid n < \omega \rangle$ and $\langle B_n \mid n < \omega \rangle$ such that for every A, B there are n_A, m_B such that for every $t_1 \in X_1$ and every $t_2 \in X_2$, $A \setminus A_{n_A} \not\subseteq t_1$ and $B \setminus B_{m_B} \not\subseteq t_2$. Consider the sequence $\langle \bigcap_{k \leq n} A_k \cap \bigcap_{k \leq n} B_k \mid n < \omega \rangle$. Then there is $A \in F$ such that for every l there is $t_l \in X_1 \cup X_2$ for which $A \setminus \bigcap_{k \leq l} A_k \cap \bigcap_{k \leq l} B_k \subseteq t_l$. For A, there are suitable n_A, m_A as above and fix $N = \max(n_A, m_A)$. Without loss of generality, $t_N \in X_1$, in which case, we have $A \setminus A_n \subseteq A \setminus \bigcap_{k \leq N} A_k \cap \bigcap_{k \leq M} B_k \subseteq t_N$, contradicting the choice of n_A. □

Note that we cannot ensure that either X_1 or X_2 contains a downward closed subset. Our next result investigates how the I-p.i.p is preserved under sums of ideals and ultrafilters.

Let I be an ideal on X and for each $x \in X$ let J_x be ideals on Y_x (resp.). We define the Fubini sum of the ideals $\sum_I J_x$ over $\bigcup_{x \in X} \{x\} \times Y_x$: For $A \subseteq \bigcup_{x \in X} \{x\} \times Y_x$,

$$A \in \sum_I J_x \text{ iff } \{x \in X \mid (A)_x \notin J_x\} \in I.$$

We denote by $I \otimes J = \sum_I J$. When I is an ideal on ω, we define transifinitely for $\alpha < \omega_1$ $I^{\otimes \alpha}$. $I^{\otimes 1} = I$, at the successor step $I^{\otimes(\alpha+1)} = I^{\otimes \alpha} \times I$. At limit step α, we fix some cofinal sequence $\langle \alpha_n \mid n < \omega \rangle$ unbounded in α and define $I^\alpha = \sum_I I^{\otimes \alpha_n}$. The above definition of Fubini sum is nothing but the dual operation of the Fubini sum of filters:

Fact 2.11. $(\sum_I J_x)^* = \sum_{I^*} J_x^*$ and in particular $(I \otimes J)^* = I^* \cdot J^*$.

Proposition 2.12. *Let F, F_n be filters over countable sets. Suppose that $I \subseteq F^*$ and $J_n \subseteq F_n^*$ are ideals for every $n < \omega$. Then if F has I-p.i.p and for every $n < \omega$, F_n has J_n-p.i.p, then $\sum_F F_n$ has $\sum_I J_n$-p.i.p.*

Proof. Let $\langle A_n \mid n < \omega\rangle$ be a sequence in $\sum_F F_n$. For each n, let $X_n = \{m < \omega \mid (A_n)_m \in F_m\} \in F$. We find $X \in F$ such that for every $n < \omega$, $X \setminus X_n \in I$. For each $m \in X$, we consider $E_m = \{n < \omega \mid m \in X_n\}$. If E_m is finite, we let $Y_m = \bigcap_{n \in E_m}(A_n)_m \in F_m$ (if E_m is empty, we let $Y_m = \omega$). Otherwise, we find $Y_m \in F_m$ such that for all $n \in E_m$, $Y_m \setminus (A_n)_m \in J_m$. Let $A = \bigcup_{m \in X}\{m\} \times Y_m$. Then Clearly, $A \in \sum_F F_n$. Let $n < \omega$ and consider $A \setminus A_n$. If

$$A \setminus A_n = (\bigcup_{x \in X \cap X_n} \{x\} \times Y_x \setminus (A_n)_x) \cup (\bigcup_{x \in X \setminus X_n} \{x\} \times Y_x).$$

If $x \notin X \setminus X_n$ then $(A \setminus A_n)_x = Y_x \setminus (A_n)_x$. Since $x \in X_n$, we have $n \in E_x$ and therefore $Y_x \setminus (A_n)_x \in J_x$. We conclude that

$$\{x < \omega \mid (A \setminus A_n)_x \notin J_n\} = X \setminus X_n \in I.$$

Hence $A \setminus A_n \in \sum_I J_n$. □

One way to obtain non-trivial sets T for which an ultrafilter U has the T-p.i.p is by intersecting U with another ultrafilter:

Theorem 2.13. *Suppose that $U_1, U_2, ...U_n$ are any ultrafilters, then for each $1 \leq i \leq n$, U_i have the $(U_1 \cap U_2 \cap ... \cap U_n)^*$-p.i.p. In particular $U_i^\omega \leq_T U_i \times (U_1 \cap U_2 \cap ... \cap U_n)^{*\omega}$.*

Proof. Fix any $1 \leq i \leq n$. Note that $(\bigcap_{j=1}^n U_j)^* = \bigcap_{j=1}^n (U_j)^*$. Suppose otherwise, then there is a sequence $\langle X_n \mid n < \omega\rangle \subseteq U_i$ such that for every $X \in U_i$ there is $n < \omega$ such that $X \setminus X_n \notin \bigcap_{j=1}^n U_j^*$. Since $X \setminus X_n \in U_i^*$, this means that there is $j \neq i$ such that $X \setminus X_n \notin U_j^*$. Since U_j is an ultrafilter, it follows that $X \setminus X_n \in U_j$ and therefore $X \in U_j$. We conclude that $U_i \subseteq \bigcup_{j \neq i} U_j$. There must be $j \neq i$ such that $U_i \subseteq U_j$, just otherwise, for each $j \neq i$ find $X_j \in U_i$ such that $X_j \notin U_j$. Then $X^* = \bigcap_{j \neq i} X_j \in U_i$. But then there is $j' \neq i$ such that $X^* \in U_{j'}$. It follows that $X_{j'} \in U_{j'}$, contradicting our choice of $X_{j'}$. Since ultrafilters are maximal with respect to inclusion, and $U_i \subseteq U_j$, we conclude that $U_i = U_j$. Now this is again a contradiction since for some (any) X, X_n and $X \setminus X_n$ disjoint and both in U. □

The argument above can be generalized to an infinite set of ultrafilters in some cases. Recall that U is an accumulation point (in the topological space $\beta\omega \setminus \omega$) of a set of ultrafilters $\mathcal{A} \subseteq \beta\omega \setminus \omega$ if and only if $U \subseteq \bigcup \mathcal{A} \setminus \{U\}$.

Proposition 2.14. *Suppose that U is not an accumulation point of $\mathcal{A} \subseteq \beta\omega \setminus \omega$. Then U has the $(\bigcap \mathcal{A})^*$-p.i.p. In particular $U^\omega \leq_T U \times (\bigcap \mathcal{A})^\omega$.*

Proof. Otherwise, we get that for some sequence $\langle X_n \mid n<\omega\rangle \subseteq U$, for every $X\in U$, there is n such that $X\setminus X_n\in V$ for some $V\in\mathcal{A}$. Since $X\setminus X_n\notin U$, it follows $V\neq U$. It follows that $U\subseteq \bigcup\mathcal{A}\setminus\{U\}$, contradiction. □

Recall that a sequence $\langle U_n \mid n<\omega\rangle$ of ultrafilters on ω is called *discrete* if there are disjoint sets $A_n\in U_n$. This is just equivalent to being a discrete set in the space $\beta\omega\setminus\omega$. In particular, no point U_n is in the closure of the others.

Corollary 2.15. *Suppose that U is not in the closure of the U_n's, namely, there is a set $A\in U$ such that for every n, $A\notin U_n$, then U has the $\bigcap_{n<\omega} U_n^*$-p.i.p.*

Corollary 2.16. *If U_n is discrete then each U_n has the $\bigcap_{n<\omega} U_n^*$-p.i.p.*

The partition given by the discretizing sets of a discrete sequence of ultrafilters can be used to compute the cofinal type of the filter obtained by intersecting the sequence.

Proposition 2.17. *Suppose that U_n is discrete. Then $\bigcap_{n<\omega} U_n \equiv_T \prod_{n<\omega} U_n$.*

Proof. Let A_n be a partition of ω so that $A_n\in U_n$. Define $f(\langle B_n \mid n<\omega\rangle)=\bigcup_{n<\omega} B_n\cap A_n$. It is clearly monotone. If $X\in\bigcap_{n<\omega}U_n$, we let $X\cap A_n=B_n\in U_n$ Then $X=X\cap\omega=X\cap(\bigcup_{n<\omega}A_n)=\bigcup_{n<\omega}X\cap A_n=f(\langle B_n\mid n<\omega\rangle)$. To see it is unbounded, suppose that $f''\mathcal{A}$ is bounded by $B\in\bigcap_{n<\omega}U_n$, then for every $A\in\mathcal{A}$, $A\cap A_n\supseteq B\cap A_n$. and therefore $\langle B\cap A_n \mid <\omega\rangle$ would bound $\mathcal{A}$. □

Corollary 2.18. *If $\langle U_n \mid n<\omega\rangle$ is a discrete sequence of ultrafilters then for every $X\subseteq Y$, $\bigcap_{n\in X}U_n\leq_T\bigcap_{m\in Y}U_m$*

2.2. Simple and deterministic ideals.

Definition 2.19. An ideal I is *simple* if for every ideal J, $I\subseteq J$, $I\leq_T J$.

Clearly any ultrafilter is simple, also fin is simple as fin $\equiv_T \omega\leq_T F$ for any filter F which is non-principal. To construct examples of ideals which are not simple, we have the following lemma.

Lemma 2.20. *Suppose that $\langle U_n \mid n<\omega\rangle$ is a sequence of discrete ultrafilters, $X\subseteq\omega$ and $n<\omega$ such that $U_n\not\leq_T\prod_{m\neq n}U_m$, then for every $X\cup\{n\}\subseteq Y$, $\bigcap_{n\in X}U_n<_T\bigcap_{m\in Y}U_m$*

Proof. $\bigcap_{n\in X}U_n\leq_T\bigcap_{m\in Y}U_m$ follows from Corollary 2.18. To see that it is strict, suppose otherwise, and let $t\in Y\setminus X$, then $U_t\leq_T\prod_{m\in Y}U_m$ but by assumption $U_t\not\leq_T\prod_{n\in X}$ and therefore $\bigcap_{n\in X}U_n\not\equiv_T\bigcap_{m\in Y}U_m$. □

Clearly, taking any $X\cup\{n\}\subseteq Y$ in the previous lemma, we get $\bigcap_{m\in Y}U_m\subseteq\bigcap_{n\in X}U_n$. Hence $\bigcap_{m\in Y}U_m$ is not simple.

Remark 2.21. A sequence $\langle U_n\mid n<\omega\rangle$ of ω-many mutually generic ultrafilters for $P(\omega)/\text{fin}$ would be a discrete sequence satisfying the assumptions of

Lemma 2.20. To see this, note that by Lemma 1.22 $U_0\times\prod_{1<m<\omega}U_n$ is Tukey equivalent to some basically generated ultrafilter in $V[U_0,\langle U_m\mid m>1\rangle]$. Now we argue as in Lemma 1.22, concluding that $U_1\not\leq_T U_0\times\prod_{1<m<\omega}U_m$.

Definition 2.22. We say that an ideal I is *deterministic* if there is a cofinal set $\mathcal{B}\subseteq I$ such that for every $\mathcal{A}\subseteq\mathcal{B}$, $\bigcup\mathcal{A}\in I$ or $\bigcup\mathcal{A}\in I^*$.

Example 2.23. We claim that fin is deterministic. Indeed, let $\mathcal{B}=\omega$. Then clearly, $\mathcal{B}$ is a cofinal in fin. Suppose that $\mathcal{A}\subseteq\omega$ is such that $\bigcup\mathcal{A}\notin fin$, then $\mathcal{A}$ is an unbounded set of natural numbers and therefore $\bigcup\mathcal{A}=\omega\in\text{fin}^*$. We will see later that for every α, $\text{fin}^{\otimes\alpha}$ is deterministic.

The reason that deterministic ideals are interesting is due to the following proposition:

Proposition 2.24. *If I is deterministic then I is simple.*

Proof. Let $I\subseteq J$ and let $\mathcal{B}\subseteq I$ be the cofinal set witnessing that I is deterministic. Let us prove that the identity function $id:\mathcal{B}\to J$ is unbounded. Suppose that $\mathcal{A}\subseteq\mathcal{B}$ is unbounded, then $\bigcup\mathcal{A}\notin I$, since otherwise, as $\mathcal{B}$ is cofinal in I, there would have been $b\in\mathcal{B}$ bounding $\mathcal{A}$. By definition of deterministic ideals, it follows that $\bigcup\mathcal{A}\in I^*$, and since $I\subseteq J$, $I^*\subseteq J^*$ hence $\bigcup\mathcal{A}\in J^*$. We conclude that $\bigcup\mathcal{A}\notin J$, namely, $\mathcal{A}$ is unbounded in J. Hence the identity function witnesses that $I\equiv_T\mathcal{B}\leq_T J$. □

Proposition 2.25. *Suppose that $I\subseteq X$ is a deterministic ideal over X.*

1. *If $\pi:X\to Y$ is injective on a set in I. Then*

$$\pi_*(I):=\{a\mid\pi^{-1}[a]\in I\}$$

is deterministic.

2. *If $A\subseteq X$, then $I\cap P(A)$ is deterministic.*

Proof. For (1), let $\mathcal{B}\subseteq I$ be a witnessing cofinal set. Let $\mathcal{C}=\{(Y\setminus(\pi''[X\setminus b]))\mid b\in\mathcal{B}\}$. Then $\mathcal{C}$ is a cofinal set in $\pi_*(I)$. Indeed, if $A\in\pi_*(I)$, then $\pi^{-1}[A]\in I$ and there is $b\in\mathcal{B}$ such that $b\supseteq\pi^{-1}[A]$. If $y\notin Y\setminus(\pi''X\setminus b)$, then $y=\pi(x)$ for some $x\in X\setminus b$. Since $\pi^{-1}[A]\subseteq b$, then $x\notin\pi^{-1}[A]$ which then implies that $y=\pi(x)\notin A$. We conclude that $A\subseteq Y\setminus(\pi''[X\setminus b])\in\mathcal{C}$, as wanted. We claim that $\mathcal{C}$ witnesses that $\pi_*(I)$ is deterministic. Let $\mathcal{A}\subseteq\mathcal{B}$ be such that $\bigcup_{a\in\mathcal{A}}Y\setminus(\pi''[X\setminus a])\notin\pi_*(I)$. Then $\pi^{-1}[\bigcup_{a\in\mathcal{A}}Y\setminus(\pi''[X\setminus a])]\notin I$. Simplifying the above set we have

$$\pi^{-1}[\bigcup_{a\in\mathcal{A}}Y\setminus(\pi''[X\setminus a])]=\pi^{-1}[Y\setminus(\bigcap_{a\in\mathcal{A}}(\pi''[X\setminus a]))]=X\setminus\bigcap_{a\in\mathcal{A}}\pi^{-1}[\pi''[X\setminus a]]$$

$$=\bigcup_{a\in\mathcal{A}}X\setminus\pi^{-1}[\pi''[X\setminus a]]=\bigcup_{a\in\mathcal{A}}a$$

The last inclusion holds as for each a, $X\setminus a=\pi^{-1}[\pi''[X\setminus a]]$ as π is one-to-one. Then $a=X\setminus(X\setminus a)=X\setminus\pi^{-1}[\pi''[X\setminus a]]$. It follows that $\bigcup_{a\in\mathcal{A}}a\notin I$.

Since $\mathcal{A} \subseteq \mathcal{B}$, we conclude that

$$\pi^{-1}[\bigcup_{a\in\mathcal{A}} Y \setminus (\pi''[X \setminus a])] = \bigcup_{a\in\mathcal{A}} a \in I^*.$$

Namely, $\bigcup_{a\in\mathcal{A}} Y \setminus (\pi''[X \setminus a]) \in \pi_*(I^*) = \pi_*(I)^*$.

For (2), again, let $\mathcal{B} \subseteq I$ be a cofinal set witnessing that I is deterministic. Consider $\mathcal{C} = \{b \cap A \mid b \in \mathcal{B}\}$. Then $\mathcal{C}$ is cofinal in $I \cap P(A)$. If $\bigcup_{b\in\mathcal{A}} b \cap A \notin I \cap P(A)$, then $\bigcup_{b\in\mathcal{A}} b \cap A \notin I$ (as it is clearly in $P(A)$). It follows that $\bigcup_{b\in\mathcal{A}} b \notin I$ and since I is deterministic, $\bigcup_{b\in\mathcal{A}} b \in I^*$.

It follows that $\bigcap_{b\in\mathcal{A}} X \setminus b \in I$ and $A \cap \bigcap_{b\in\mathcal{A}} X \setminus b \in I \cap P(A)$. But

$$A \cap \bigcap_{b\in\mathcal{A}} X \setminus b = \bigcap_{b\in\mathcal{A}} A \setminus (b \cap A) = A \setminus \bigcup_{b\in\mathcal{A}} b \cap A.$$

Hence $\bigcup_{b\in\mathcal{A}} b \cap A \in (I \cap P(A))^*$. □

Note that (2) above can be vacuous if $A \in I$, since in that case $I \cap P(A)$ is not proper. So we should at least assume that $A \in I^+$. Generally speaking, it is unclear whether an ideal relative to a positive set has the same Tukey-type. However, if the ideal is deterministic, this type does not change:

Fact 2.26. Suppose that I is an ideal over X and $A \in I^+$, then $I \equiv_T I \cap P(A)$.

Proof. Let $\mathcal{B}$ be a witnessing cofinal set for I, and let $f : \mathcal{B} \to I \cap P(A)$ be the map $f(b) = b \cap A$. Then clearly, the map is monotone and its image is the cofinal set $\mathcal{C}$ from the proof of (2) from the previous proposition (and therefore cofinal). To see it is unbounded, suppose that $\bigcup \mathcal{A} \notin I$, then $\bigcup \mathcal{A} \in I^*$ and the computation from the previous proposition applies to show that $\bigcup f''\mathcal{A} \in (I \cap P(A))^*$ and in particular not in I. Hence f is unbounded. □

Corollary 2.27. *Suppose that $I^\omega \equiv_T I$, I is deterministic. Then for every ultrafilter U such that $I \subseteq U^*$ and U satisfies the I-p.i.p, $U \cdot U \equiv_T U$.*

Proof. Since I is deterministic, $I^\omega \equiv_T I \leq_T U$. Since U has the I-p.i.p we can apply Theorem 2.6 to conclude that $U \cdot U \equiv_T U$. □

Given any $\{X_i \mid i \in N\}$, where each $X_i \subseteq P(\omega)$, there is the smallest ideal (might not be proper) that contains all the X_i. We denote this ideal by $I(\{X_i \mid i \in N\})$. It is generated by the sets $\{\bigcup_{i\in M} b_i \mid b_i \in X_i,\ M \in [N]^{<\omega}\}$. We can replace each X_i be some cofinal set in B_i and obtain the same generated ideal.

Theorem 2.28. *Suppose that I is an ideal $I = I(\langle I_n \mid n < \omega\rangle)$, where $I_n \subseteq I$ are deterministic ideals. Then I is deterministic.*

Proof. Let $\mathcal{B}_n \subseteq I_n$ be a cofinal set witnessing I_n being deterministic. Let $\mathcal{B}'_n = \{b \cup n \mid b \in \mathcal{B}_n\}$. Then $I = I(\langle \mathcal{B}_n \mid n < \omega\rangle)$. Consider the cofinal set $\mathcal{B}$ of all sets of the form $X_{T,(b'_i)_{i\in T}} := \bigcup_{n\in T} b'_n$ where $b'_n \in \mathcal{B}'_n$ and $T \in [\omega]^{<\omega}$. Then $\mathcal{B}$ is a cofinal set in I. Suppose $\bigcup_{j\in S} X_{T_j,(b^j_i)_{i\in T_j}} \notin I$.

Note that $X_{T,(b'_i)_{i\in T}} \supseteq \max(T)$. And so, if $\bigcup_{j\in S} T_j$ is unbounded in ω, then $\bigcup_{j\in S} X_{T_j,(b^j_i)_{i\in T_j}} = \omega \in I^*$. Otherwise, $\bigcup_{j\in S} T_j$ is bounded by some N and therefore

$$\bigcup_{j\in S} X_{T_j,(b^j_i)_{i\in T_j}} \subseteq \{0,...,N\} \cup \bigcup_{i\leq N}(\bigcup_{b\in\mathcal{A}_i} b),$$

where $\mathcal{A}_i = \{b^i_j \mid i \in T_j\} \subseteq \mathcal{B}_i$. Since this is a finite union of sets which is not in I, there is $i \leq N$ such that $\bigcup_{b\in\mathcal{A}_i} b \notin I$, and in particular, $\bigcup_{b\in\mathcal{A}_i} b \notin I_i$. Since $\mathcal{B}_i$ is a witness for I_i being deterministic, $\bigcup_{b\in\mathcal{A}_i} b \in I^*_i \subseteq I^*$. Since $\bigcup_{b\in\mathcal{A}_i} b \subseteq \bigcup_{j\in S} X_{T_j,(b^j_i)_{i\in T_j}}$, it follows that $\bigcup_{j\in S} X_{T_j,(b^j_i)_{i\in T_j}} \in I^*$. $\square$

Proposition 2.29. *Suppose that* $\text{fin} \subseteq I$ *be any ideal over* ω*, and* $\langle J_n \mid n < \omega\rangle$ *is a sequence of deterministic ideals over* ω *such that for every* $n < \omega$, $J_{n+1} \geq_T J_n$*. Then* $\sum_I J_n$ *is deterministic.*

Proof. Let $\mathcal{B}_n \subseteq J_n$ witness that J_n is deterministic. Let $\langle f_{m,n} : J_n \to J_m \mid n \leq m < \omega\rangle$ be a sequence of unbounded maps. Denote by $\mathcal{B}$ the set of all sequences $\vec{b} = \langle b_n \mid n < \omega\rangle \in \prod_{n<\omega} \mathcal{B}_n$ such that for every $n < m < \omega$, $b_m \supseteq f_{m,n}(b_n)$. For $\vec{b} \in \mathcal{B}$ and $A \in I$ defined

$$C_{A,\vec{b}} = \bigcup_{n\in A}\{n\} \times \omega \cup \bigcup_{n\notin A}\{n\} \times b_n.$$

Let us show that $\mathcal{C} = \{C_{A,\vec{b}} \mid A \in I, \vec{b} \in \mathcal{B}\}$ is cofinal in $\sum_I J_n$. Let $Z \in \sum_I J_n$, and $A = \{n \mid (Z)_n \notin J_n\}$. By definition of $\sum_I J_n$, $A \in I$. Construct an increasing sequence $\langle b_n \mid n < \omega\rangle \in \prod_{n<\omega} \mathcal{B}_n$ such that for every $n \notin A$, $(Z)_n \subseteq b_n$ and for every $n < m < \omega$, $f_{m,n}(b_n) \subseteq b_m$. It is possible to construct such a sequence recursively. At stage n, we need to find b_n such that for all $k < n$, $f_{n,k}(b_k) \subseteq b_n$ and if $n \notin A$, then also $(Z)_n \subseteq b_n$. These are finitely many sets in J_n and therefore, since $\mathcal{B}_n$ is a cofinal set in J_n, we can find a single $b_n \in \mathcal{B}_n$ including the union of these sets. It follows that for every n, $(Z)_n \subseteq (C_{A,\vec{b}})_n$ and therefore $Z \subseteq C_{A,\vec{b}}$. Let us prove that $\mathcal{C}$ witnesses that $\sum_I J_n$ is deterministic. Suppose that $\bigcup_{i\in T} C_{X_i,\vec{b}^i} \notin \sum_I J_n$. Then $A := \{n < \omega \mid (\bigcup_{i\in T} C_{X_i,\vec{b}^i})_n \notin J_n\} \notin I$. Note that $(\bigcup_{i\in T} C_{X_i,\vec{b}^i})_n = \bigcup_{i\in T}(C_{X_i,\vec{b}^i})_n$. If there is $i_0 \in T$ such that $n \in X_{i_0}$, then $(C_{X_{i_0},\vec{b}^{i_0}})_n = \omega$ and in particular $\omega = \bigcup_{i\in T}(C_{X_i,\vec{b}^i})_n \in J^*_n$. Otherwise, consider $n \in A$ such that for every $i \in T$, $n \notin X_i$. Then $(C_{X_i,\vec{b}^i})_n \in \mathcal{B}_n$ for every $i \in T$ and $\bigcup_{i\in T}(C_{X_i,\vec{b}^i})_n \notin J_n$. Since $\mathcal{B}_n$ witnesses that J_n is deterministic, $\bigcup_{i\in T}(C_{X_i,\vec{b}^i})_n \in J^*_n$. Fix any $n_0 \in A$, which exists [18] as $A \notin I$. For every $n \geq n_0$, either there is $i \in T$ such that $n \in X_i$, and as we have seen, $\cup_{i\in T}(C_{X_i,\vec{b}^i})_n = \omega \in J^*_n$. Otherwise, for every $i \in T$, $(C_{X_i,\vec{b}^i})_n = b^i_n \in \mathcal{B}_n$, and by the assumption, $f_{n,n_0}(b^i_{n_0}) \subseteq b^i_n$. Since $n_0 \in A$, $\bigcup_{i\in T} b^i_{n_0} \notin J_{n_0}$, and since f_{n,n_0} is unbounded, $\bigcup_{i\in T} f_{n,n_0}((C_{X_i,\vec{b}^i})_n) \notin J_n$. Since $\mathcal{B}_n$ witnesses

[18] If I is principle then we pick n_0 such that $\{n_0\} \in I^+$.

that J_n is deterministic, it follows that $\bigcup_{i\in T} f_{n,n_0}((C_{X_i,\vec{b}^i})_n)\in J_n^*$. We conclude that for every $n\geq n_0$, $\bigcup_{i\in T}(C_{X_i,\vec{b}^i})_n\in J_n^*$. Since $\{n_0,n_0+1,...\}\in I^*$, we have that $\bigcup_{i\in T} C_{X_i,\vec{b}^i}\in(\sum_I J_n)^*$ as wanted.

□

Corollary 2.30. *Suppose that I,J are ideals over ω such that J is deterministic. Then $I\otimes J$ is deterministic.*

Corollary 2.31. *For every I, $I\otimes\text{fin}$ is deterministic.*

Corollary 2.32. *For any non-principal ultrafilter U such that $U\equiv_T U\cdot U$ and for every $I\subseteq U^*$, $I^\omega\leq_T U$.*

Proof. $I\leq_T I\otimes\text{fin}\leq_T U\cdot U$. Hence by Corollary 1.8

$$I^\omega\leq_T (U\cdot U)^\omega\equiv_T (U\cdot U)\cdot(U\cdot U)\equiv_T U\cdot U\equiv_T U.$$

□

3. Proof of the main result and some corollaries

Let us turn to the proof of the main result which appears in Theorem 3.2 below.

Corollary 3.1. *For any non-principal ultrafilter U,V over ω, $V\cdot U$, $U\cdot V\geq_T (U^*\cap V^*)^\omega$*

Proof. We have that $U^*\cap V^*\subseteq V^*$ and $\text{fin}\subseteq V^*$. We conclude that $(U^*\cap V^*)\otimes\text{fin}\subseteq(V\cdot V)^*$. By Corollary 2.31, $(U^*\cap V^*)\otimes\text{fin}$ is deterministic, hence

$$U^*\cap V^*\leq_T (U^*\cap V^*)\otimes\text{fin}\leq_T V\cdot V$$

By Milovich's Corollary 1.8 and Theorem 1.7

$$U\cdot V\equiv_T U\cdot(V\cdot V)\equiv_T U\times(V\cdot V)^\omega\geq_T (U^*\cap V^*)^\omega.$$

The proof that $V\cdot U\geq_T (U^*\cap V^*)^\omega$ is symmetric. □

Theorem 3.2. *Suppose that U,V are ultrafilters on ω, then*

$$U\cdot V\equiv_T U\times V\times(U\cap V)^\omega.$$

Hence $U\cdot V\equiv_T V\cdot U$.

Proof. By the previous corollary, $(U\cap V)^\omega\equiv_T (U^*\cap V^*)^\omega\leq_T U\cdot V$. Since $U\times V\times(U\cap V)^\omega$ is the least upper bound in the Tukey order of $U,V,(U\cap V)^\omega$, we have

$$U\times V\times(U\cap V)^\omega\leq_T U\cdot V$$

For the other direction, recall that by Theorem 2.13 V has the $(U\cap V)^*$-p.i.p and therefore by Proposition 2.5

$$U\cdot V\equiv_T U\times V^\omega\leq_T U\times V\times(U^*\cap V^*)^\omega\equiv_T U\times V\times(U\cap V)^\omega.$$

□

Recall that for an ultrafilter U, $Ch(U)$ denoted the minimal size of a cofinal subset of U. It is clear that if $U \equiv_T U'$ then $Ch(U) = Ch(U')$. As a corollary of theorem 3.2, we get:

Corollary 3.3. *For any two ultrafilters,* $Ch(U \cdot V) = Ch(V \cdot U)$.

It is possible to prove by induction now that the product of n-ultrafilters commute, but we would like to get the exact cofinal type of such product. We will need the following facts:

Fact 3.4. (1) $U_1 \cdot U_2 \cdot ... \cdot U_n \equiv_T U_1 \times (U_2)^\omega ... \times (U_n)^\omega$.
(2) $(\mathbb{P}^\omega)^\omega \simeq \mathbb{P}^\omega$.

Proof. (1) is just by induction using Milovich Theorem 1.7. For (2), just decompose ω into infinitely many infinite sets. See for example [3, Fact 2.4]. $\square$

Corollary 3.5. *For every ultrafilters* $U_1, ..., U_n$*,*

$$U_1 \cdot U_2 \cdot ... \cdot U_n \equiv_T U_1 \times U_2 \times ... \times U_n \times (U_1 \cap ... \cap U_n)^\omega$$

Proof. Following the proof of Theorem 3.2, $U_1 \cap ... \cap U_n \subseteq U_n$ and we can prove similarly that $(U_1 \cap .... \cap U_n)^\omega \leq_T U_n \cdot U_n \leq_T U_1 \cdot ... \cdot U_n$. We conclude that

$$U_1 \times ... \times U_n \times (U_1 \cap ... \cap U_n)^\omega \leq_T U_1 \cdot ... \cdot U_n$$

In the other direction, each U_i has the $(U_1 \cap ... \cap U_n)^*$-p.i.p and therefore

$$U_1 \cdot ... \cdot U_n \equiv_T U_1 \times (U_2)^\omega \times (U_3)^\omega \times ... \times (U_n)^\omega \leq_T$$

$$\leq_T U_1 \times U_2 \times (U_1 \cap .... \cap U_n)^\omega \times U_3 \times (U_1 \cap ... \cap U_n)^\omega \times ... \times U_n \times (U_1 \cap ... \cap U_n)^\omega$$

$$\leq_T U_1 \times ... \times U_n \times ((U_1 \cap ... \cap U_n)^\omega)^\omega \equiv_T U_1 \times .... \times U_n \times (U_1 \cap ... \cap U_n)^\omega$$

$\square$

Corollary 3.6. *Suppose that* $U \cdot U \equiv_T U$ *then for every* $V \geq_T U$*,* $V \cdot V \equiv_T V$*. Namely the class of ultrafilters* U *such that* $U \cdot U \equiv_T U$ *is upward closed and the class of ultrafilters* V *such that* $V \cdot V >_T V$ *is downward closed.*

Proof. Otherwise, $V <_T V \cdot V$ and

$$U \cdot V \equiv_T U \times V^\omega \equiv_T V^\omega \equiv_T V \cdot V >_T V \equiv_T V \times U \equiv_T V \times U^\omega \equiv_T V \cdot U,$$

contradiction. $\square$

Corollary 3.7. *For every* α*,* $\text{fin}^{\otimes\alpha}$ *is deterministic.*

Proof. The base case $\text{fin}^{\otimes 1} = \text{fin}$ which is deterministic. Suppose this is true for α, then $\text{fin}^{\otimes\alpha+1} = \text{fin} \cdot \text{fin}^{\otimes\alpha}$ and by induction hypothesis and Corollary 2.30 $\text{fin}^{\otimes\alpha+1}$ is deterministic. At limit step, $\text{fin}^{\otimes\alpha} = \sum_{\text{fin}} \text{fin}^{\otimes\alpha_n}$. By it is known that fin^{α_n} is increasing in the Tukey order (even in the Rudin-Keisler order, see for example [3, Lemma 3.2]) and by induction hypothesis are all deterministic. Therefore, by Proposition 2.29 $\text{fin}^{\otimes\alpha}$ is deterministic.

$\square$

In [3] it was noticed that a generic ultrafilter for $P(X)/I$ where I is a σ-ideal satisfies the I-p.i.p. now together with Corollary 2.27, we recover the result from [3] in our abstract settings:

Corollary 3.8. *For every α, a generic ultrafilter G for $P(\omega^\alpha)/\text{fin}^{\otimes\alpha}$ satisfies $G \cdot G \equiv_T G$.*

Proof. G has the $\text{fin}^{\otimes\alpha}$-p.i.p, and by Corollary 3.7 $\text{fin}^{\otimes\alpha}$ is deterministic. Also note that $(\text{fin}^{\otimes\alpha})^\omega \equiv_T \text{fin}^{\otimes\alpha}$ (see [3, Thm. 3.3 & Fact 2.4]). Thus by Corollary 2.27 $G \cdot G \equiv_T G$. □

4. On ultrafilters above I^ω

The cofinal type of ω^ω came up in several papers [28, 36] regarding the Tukey order on general ultrafilters. Milovich asked whether there is an ultrafilter U such that $(U, \supseteq)$ is Tukey equivalent to ω^ω. Let us point out that a negative answer is a straightforward corollary[19] of Theorem 0.3:

Proposition 4.1. *There is no non-principal ultrafilter U over ω such that $(U, \supseteq) \equiv_T \omega^\omega$.*

Proof. By Sierpinski [35], a non-principal ultrafilter over ω is a non-measurable set as a subset of 2^ω and in particular non-analytic. An ultrafilter U with the topology inherited from 2^ω is a separable metric space and the set of predecessors is compact and ω^ω is a basic analytic order, hence by Theorem 0.3, $(U, \supseteq) \not\leq_T \omega^\omega$. □

It turns out that some general problems boil down to being Tukey above ω^ω. Such a problem is addressed in Theorem 0.4, which sets up the equivalence for p-point ultrafilter U, between $U \cdot U \equiv_T U$ and $U \geq_T \omega^\omega$. This was generalized in Theorem 0.5 which ensures that for a general ultrafilter U, $U \cdot U \equiv_T U$ is equivalent to the existence of *some* $I \subseteq U^*$ such that U has the I-p.i.p and $U \geq_T I^\omega$. In the first part of this section, we tighten the connection between the I-p.i.p and being above I^ω for a deterministic ideal I. Then, in the second subsection, we shall restrict our attention to $I = \omega$.

4.1. The case of a deterministic ideal I.

There is a slight difference between the type of equivalence in Theorem 0.4 to the equivalence $U \cdot U \equiv_T U$ for p-points and the general one in 0.5. Indeed, in the latter, the ideal I can vary. Hence it is unclear in general if for a fixed I, the following is true: for any ultrafilter U which has the I-p.i.p, $U \cdot U \equiv_T U$ iff $U \geq_T I^\omega$. Let us note first that such equivalence holds for simple ideals (and therefore also for deterministic ideals).

Proposition 4.2. *Suppose that I is simple. Then for any ultrafilter U which has the I-p.i.p, $U \cdot U \equiv_T U \times I^\omega$. Therefore, the following are equivalent:*

(1) $U \cdot U \equiv_T U$.
(2) $U \geq_T I^\omega$.

[19] Milovich's question appeared only 4 years after Solecki and Todorcevic's result.

Proof. By Theorem 1.7, $U \cdot U \equiv_T U^\omega$. Since I is simple, and $I \subseteq U^*$, $U \geq_T I$ and in particular $U \cdot U \equiv_T U^\omega \geq_T U \times I^\omega$. The other direction follows from the I-p.i.p of U and Proposition 2.5. Now to see the equivalence, (2) $\Rightarrow$ (1) follows from Theorem 2.6, and (1) $\Rightarrow$ (2) follows from the first part as $U \cdot U \equiv_T U \times I^\omega \leq_T U \leq_T U \cdot U$. $\square$

Our next objective it to study the class of ultrafilters which are Tukey above I^ω. The next theorem shows that for deterministic I's this class extends the class of ultrafilters which do not have the I-p.i.p.

Theorem 4.3. *Suppose that* $\text{fin} \subseteq I \subseteq U^*$ *is deterministic, then if* $U \not\geq_T I^\omega$, *then* U *has the* I*-p.i.p*

Proof. Let us verify the equivalent condition in Proposition 2.7, let $\langle A_n \mid n < \omega\rangle$ be a partition of ω such that $A_n \notin U$. We need to find $X \in U$ such that $X \cap A_n \in I$ for every n. Without loss of generality, suppose that $A_n \in I^+$ for every n. Since $\text{fin} \subseteq I$, A_n is infinite and we can find a bijection $\pi : \omega \leftrightarrow \omega \times \omega$ such that $\pi'' A_i = \{i\} \times \omega$. Let $W = \pi_*(U)$ be the Rudin-Keisler isomorphic copy of U. For each $n < \omega$, consider the ideal $I_n = \pi_*(I \cap P(A_n))$ on $\{n\} \times \omega$. By Proposition 2.25, $I \cap P(A_n)$ is a deterministic and since $\pi \restriction A_n$ is one-to-one $I_n = \pi_*(I \cap P(A_n))$ is deterministic. It follows by 2.29 $\sum_{\text{fin}} I_n$ is deterministic. Moreover, by Fact 2.26, $I \equiv_T I \cap P(A_n) \equiv_T I_n$ and therefore

$$I^\omega \equiv_T \prod_{n<\omega} I_n \equiv_T \sum_{\text{fin}} I_n$$

Since $U \not\geq_T I^\omega$, $W \not\geq_T \sum_{\text{fin}} I_n$. Since $\sum_{\text{fin}} I_n$ is deterministic, it follows that $\sum_{\text{fin}} I_n \not\subseteq W^*$. Thus, there is $X' \in \sum_{\text{fin}} I_n \cap W$. Namely, for all but finitely many n's, $(X')_n \in I_n$. Since each $A_i \notin U$, we may assume that for every n, $(X')_n \in I_n$. Let $X = \pi^{-1}[X']$, then for every $n < \omega$, $X \cap A_n \in I$ as $\pi'' X \cap A_n = \{x\} \cap (X)_n \in I_n$. $\square$

The proof of the above actually gives the following:

Corollary 4.4. *Suppose that* U *does not have the* I*-p.i.p, then there is* $W \subseteq \omega \times \omega$ *such that* $W \equiv_{RK} U$ *and* $\sum_{\text{fin}} I_n \subseteq W$, *and each* $I_n \equiv_T I$. *In particular, the Tukey type of* I^ω *is realized as a sub-ideal of* U^*.

Taking $I = \text{fin}$ in the above we obtain the following corollary

Corollary 4.5. *Suppose that* U *is a non-principal ultrafilter such that* $U \not\geq_T \omega^\omega$ *then* U *is a p-point.*

As a corollary, we see that in Proposition 4.2 and therefore also in Theorem 0.4, the I-p.i.p assumption is somewhat redundant.

Corollary 4.6. *If* $I \subseteq U^*$ *is deterministic then the following are equivalent:*

(1) $U \equiv_T U \cdot U$ *or* U *does not have the* I*-p.i.p.*
(2) $U \geq_T I^\omega$

4.2. **Ultrafilters above ω^ω.** As observed by Dobrinen and Todorcevic, rapid ultrafilters form a subclass of those which are Tukey above ω^ω. By Miller [26], in the Laver model where the Borel conjecture holds[20], there are no rapid ultrafilters. On the other hand, there are always ultrafilters which are above ω^ω, namely tukey-top ultrafilters. The question regarding the existence (in ZFC) of non-Tukey top ultrafilters is a major open problem. Hence we do not expect to have a simple ZFC construction of a non-Tukey-top ultrafilter which is above ω^ω.

The result from the previous section entails some drastic consistency results regarding the class of ultrafilters above ω^ω:

Corollary 4.7. *Suppose that there are no p-points, then every ultrafilter is above ω^ω.*

Models with no p-points were first constructed by Shelah [34] and later by Chudonsky and Guzman [11].

By yet another result of Shelah, in the Miller model [27], which is obtained by countable support iteration of the superperfect tree forcing of length ω_2 over a model of CH, every p-point is generated by $\aleph_1$-many sets. It is known that $\mathfrak{d} = \mathfrak{c}$ holds in that model. Therefore, every p-point is generated by less than $\mathfrak{d}$-many sets and in particular not above ω^ω.

Corollary 4.8. *It is consistent that p-points are characterized by not being above ω^ω.*

Focusing on models less extreme than the ones above, we may be interested in those p-point which are above ω^ω. The purpose of this section is to address the question raised in [3] whether rapid p-point are exactly those p-points which are above ω^ω (The dashed area in Figure 4.2). As a warm-up, let us note that there are always non-rapid ultrafilters above ω^ω. To see this, we need the following result [26, Thm. 4]:

Proposition 4.9 (Miller)**.** *For any two ultrafilters U, V on ω, $U \cdot V$ is rapid iff V is rapid.*

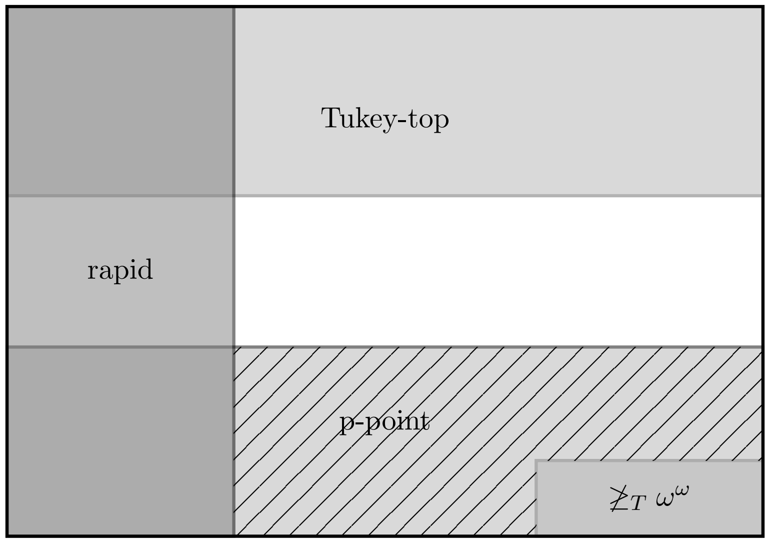
By results of Choquet [12], there are always non-rapid ultrafilters. Taking any such U, $U \cdot U$ is certainly above ω^ω and by the above result of Miller, it is non-rapid.

Corollary 4.10. *There is a non-rapid ultrafilter which is Tukey above ω^ω.*

Note that the ultrafilter we constructed in the previous proof in not a p-point as it is a product.

Figure 4.2.

[20] i.e. the model obtained by adding ω_2-many Laver reals to a model of CH.



The real issue is to construct a p-point which is not rapid but still above ω^ω. To do that, let us introduce the class of α-almost rapid ultrafilters.

Given a function $f : \omega \to \omega \setminus \{0\}$ such that $f(0) > 0$. We denote by $exp(f)(0) = f(0)$ and

$$exp(f)\big(n+1\big) = f\big(exp(f)(n)\big) = f(f(f(f...f(0)..))).$$

We define the n^{th} f-exponent function,

$$exp_0(f) = f \text{ and } exp_n(f) = exp(exp_{n-1}(f)).$$

This definition continuous transfinitely for every $\alpha < \omega_1$:

$$exp_{\alpha+1}(f) = exp(exp_\alpha(f)).$$

For limit $\delta < \omega_1$, we fix some increasing cofinal sequence $\langle \delta_n \mid n < \omega \rangle$ in δ, and let

$$exp_\delta(f)(n) = \max\{exp_{\delta_n}(f)(n), exp_\delta(f)(n-1)+1\}.$$

Lemma 4.11. *Let $f, g : \omega \to \omega$ be increasing functions.*

(1) For every $\alpha < \omega_1$, $exp_\alpha(f)$ is increasing.
(2) If $f \leq g$ then for every $\alpha < \omega_1$, $exp_\alpha(f) \leq exp_\alpha(g)$.
(3) For every $\alpha < \beta < \omega_1$, $exp_\alpha(f) <^* exp_\beta(f)$.

Proof. For (1), we proceed by induction. For $\alpha = 0$, $exp_0(f) = f$ is increasing. Suppose $exp_\alpha(f)$ is increasing, then for every $n < \omega$, $exp_\alpha(f)(n) > n$. For $\alpha + 1$, let $n < \omega$. Since $exp_\alpha(f)$ is increasing,

$$exp_{\alpha+1}(f)(n+1) = exp_\alpha(f)(exp_{\alpha+1}(f)(n)) > exp_{\alpha+1}(f)(n).$$

For limit δ, is clear from the definition that $exp_\delta(f)$ is increasing. Also (2) is proven by induction. The base case is $exp_0(f) = f \leq g = exp_0(g)$. Suppose this was true for α, and let us prove by induction on $n<\omega$. The base again is

$$exp_{\alpha+1}(f)(0)=exp_\alpha(f)(0)\leq exp_\alpha(g)(0)\leq exp_{\alpha+1}(g)(0)$$

Suppose that $exp_{\alpha+1}(f)(n)\leq exp_{\alpha+1}(g)(n)$, then by (1) and the induction hypothesis

$$\begin{aligned}exp_{\alpha+1}(f)(n+1)=exp_\alpha(f)(exp_{\alpha+1}(f)(n))\leq exp_\alpha(f)(exp_{\alpha+1}(g)(n))\\ \leq exp_\alpha(g)(exp_{\alpha+1}(g)(n))=exp_{\alpha+1}(g)(n+1)\end{aligned}$$

At limit stages δ, by the induction hypothesis,

$$exp_\delta(f)(n)=exp_{\delta_n}(f)(n)\leq exp_{\delta_n}(g)(n)=exp_\delta(g)(n).$$

Finally, (3) is a standard diagonalization argument. □

Definition 4.12. For $\alpha<\omega_1$, we say that an ultrafilter U is *α-almost-rapid* if for every function $f\in\omega^\omega$ there is $X\in U$ such that $exp_\alpha(f_X)\geq^* f$, where f_X is the increasing enumeration of X.

Remark 4.13. By strengthening the above definition, we may require that $exp_\alpha(f_X)\geq f$. However, this strengthening turns out to be an equivalent definition.

Note that 0-almost-rapid is just rapid, and by (3) of the previous lemma, if $\beta\leq\alpha$ then β-almost-rapid implies α-almost-rapid. We call U *almost-rapid* if it is 1-almost-rapid.

Proposition 4.14. *If U is α-almost-rapid implies $U\geq_T\omega^\omega$*

Proof. Consider the map $X\mapsto exp_\alpha(f_X)$. We claim that it is monotone and cofinal. First, suppose that $X\subseteq Y$, then the natural enumerations f_X,f_Y of X,Y (resp.) satisfy $f_X\geq f_Y$. Then by Lemma 4.11(3) $exp_\alpha(f_X)\geq exp_\alpha(f_Y)$. The map above is cofinal by the α-almost rapidness of U. □

Rapid ultrafilters are characterized by the following property [26]: An ultrafilter over ω is rapid if and only if for every sequence $\langle P_n\mid n<\omega\rangle$ of finite subsets of ω, there is $X\in U$ such that for every $n<\omega$, $|X\cap P_n|\leq n$. The proposition below provides an analogous characterization of almost-rapid ultrafilters.

Proposition 4.15. *The following are equivalent:*

1. *U is almost-rapid.*
2. *For any sequence $\langle P_n\mid n<\omega\rangle$ of sets, such that P_n is finite, there is $X\in U$ such that for each $n<\omega$, $exp(f_X)(n-1)\geq|X\cap P_n|$ (where $exp(f_X)(-1)=0$).*

Proof. (1) ⇒ (2): Suppose that U is almost rapid, and let $\langle P_n\mid n<\omega\rangle$ be a sequence as above. Let $f(n)=\max(P_n)+1$. By (1), there is $X\in U$ which is obtained by almost-rapidness, namely $exp(f_X)(0)=f_X(0)=\min(X)>f(0)>\max(P_0)$ and therefore $X\cap P_0=\emptyset$. Next, $exp(f_X)(1)=f_X(f_X(0))>f(1)>\max(P_1)$ hence $|P_1\cap X|\leq f_X(0)=exp(f_X)(0)$. In general $f_X(exp(f_X)(n)) = exp(f_X)(n+1) > f(n+1) > \max(P_{n+1})$ and therefore $|X \cap P_{n+1}| \leq exp(f_X)(n)$.

(2) $\Rightarrow$ (1): Let f be any function. Let $P_n = f(n)$. Then by (2), there is X such that $|X \cap f(n)| \leq exp(f_X)(n-1)$. In particular, $X \cap f(0) = \emptyset$ and therefore $exp(f_X)(0) = f_X(0) = \min(X) \geq f(0)$. In general, $|X \cap f(n)| \leq exp(f_X)(n-1)$ and therefore $f(n) < f_X(exp(f_X)(n-1)) = exp(f_X)(n)$. □

Theorem 4.16. *Assume CH. Then there is a p-point which is almost-rapid but not rapid*

Proof. Let $P_n = \{1, ..., 2^n\}$. Let $I = \{A \subseteq \omega \mid \exists k \forall n,\ |A \cap P_n| \leq k \cdot n\}$. Then I is a proper ideal on ω. Suppose that $\langle P_n \mid n < \omega \rangle$ is not a counterexample for U being rapid, then there is a set $X \in U$ such that $|X \cap P_n| \leq n$ for every n and therefore $X \in I$. Hence, as long as we have $U \subseteq I^+$, U will not be rapid. Note that

$$A \in I^+ \text{ iff for every } k, \text{ there is } n_k \text{ such that } |A \cap P_{n_k}| > kn_k.$$

Or equivalently, $n \mapsto |A \cap P_n|$ is not asymptotically bounded by a linear function of n. The following is the key lemma for our construction:

Lemma 4.17. *Suppose that $\langle A_n \mid n < \omega \rangle \subseteq I^+$ is $\subseteq$-decreasing, and $f : \omega \to \omega$. Then there is $B \subseteq \omega$ such that*

(1) $B \subseteq^* A_n$ for every n.
(2) $B \in I^+$.
(3) $exp(f_B) > f$.

Proof. Suppose without loss of generality that f is increasing. In particular, $f(k) \geq k$. Consider $f(1)$, find $2 < n_1$ so that

$$|A_1 \cap P_{n_1}| > (f(1) + 2) \cdot n_1$$

such an n_1 exists as A_1 is positive and taking $k = f(1)+2$. Find $a_0, ..., a_{n_1+1}$ such that

(1) $f(0), n_1 + 1 < a_0 < a_1 < .... < a_{n_1+1}$.
(2) $a_{n_1+1} > f(1)$.
(3) $a_0, a_1, ..., a_{n_1+1} \in A_1 \cap P_{n_1}$.

It is possible to find such elements as

$$|A_1 \cap P_{n_1} \setminus \{0, ..., n_1+1\}| > (f(1)+2)n_1 - (n_1+2) \geq 3n_1 - n_1 - 2 = 2n_1 - 2 \geq n_1 + 1.$$

So there are n_1+1 elements in $A_1 \cap P_{n_1}$ greater than n_1+1. Since $|A_1 \cap P_{n_1}| > f(1)$, we can also make sure that the n_1+1 element we choose is above $f(1)$. This way, we have guaranteed that:

(1) $f(0) < f(1) < a_0$.
(2) a_{a_0} was not defined yet (!), but as long as the sequence is increasing, $a_{a_0} > f(1)$.
(3) For $k = 1$, there is n_1 such that $|B \cap P_{n_1}| > n_1$

Now consider $f(2)$ and find $n_2 > 2, a_{n_1+1}$ so that

$$|A_2 \cap P_{n_2}| > (f(2)+2)(a_{n_1+1}+1)n_2,$$

we find

(1) $n_1 + 1 + 2n_2 + 1 < a_{n_1+2} < ... < a_{n_1+1+2n_2+1}$.
(2) $f(2) < a_{n_1+1+2n_2+1}$.
(3) $a_{n_1+2}, ..., a_{n_1+1+2n_2+1} \in A_2 \cap P_2$.

This is possible to do since

$$|A_2 \cap P_{n_2} \setminus \{0, ..., n_1 + 2n_2 + 2\}| > (f(2)+2)(a_{n_1+1}+2)n_2 - (n_1 + 2n_2 + 3) >$$

$$> 8n_2 - (3n_2 + 3) = 6n_2 - 3 > 2n_2 + 1$$

So we can find $a_{n_1+2}, ..., a_{n_1+1+2n_2+1}$ above $n_1 + 1 + 2n_2 + 1$ (and therefore also above a_{n_1+1}). We can also make sure that the last element we pick is above $f(2)$. This way we ensured the following:

(1) As we observed, a_{a_0} was not defined in the first round (and might not be defined in the second round as well) and therefore (a possibly future) $a'_1 := a_{a_0} > n_1 + 1 + 2n_2 + 1$. Thus a future $a'_2 := a_{a_{a_0}} > a_{n_1+1+2n_2+1} > f(2)$.
(2) For $k = 2$, there is n_2 such that $|B \cap P_{n_2}| > 2n_2$.

In general suppose we have defined $n_1 < n_2 < .... < n_k$ and $a_0, ..., a_{\sum_{i=1}^k in_i+1}$ and such that $a'_{k-1} > \sum_{i=1}^k in_i + 1$. Then we find $n_{k+1} > k+1, a_{\sum_{i=1}^k in_i+1}$ such that $|A_{k+1} \cap P_{n_{k+1}}| > 3(k+1)(f(k+1)+1)n_{k+1}$. We now define

$$a_{(\sum_{i=1}^k in_i+1)+1}, a_{(\sum_{i=1}^k in_i+1)+2} ..., a_{\sum_{i=1}^{k+1} in_i+1}$$

(that is $(k+1)n_{k+1} + 1$ many elements) so that:

(1) $\sum_{i=1}^{k+1} in_i + 1 < a_{(\sum_{i=1}^k in_i+1)+1} < ... < a_{\sum_{i=1}^{k+1} in_i+1}$,
(2) $a_{\sum_{i=1}^{k+1} in_i+1} > f(k+1)$.
(3) $a_{(\sum_{i=1}^k in_i+1)+1}, ..., a_{\sum_{i=1}^{k+1} in_i+1} \in A_{k+1} \cap P_{n_{k+1}}$.

To see that such a's exists, note that

$$|A_{k+1} \cap P_{n_{k+1}} \setminus \{0, ..., \sum_{i=1}^{k+1} in_i + 1\}| > 3(k+1)(f(k+1)+1)n_{k+1} - (\sum_{i=1}^{k+1} in_i + 1) - 1$$

$$> 3(k+1)(f(k+1)+1)n_{k+1} - ((k+1)n_{k+1} + 1) - (\sum_{i=1}^{k} in_i + 1) - 1$$

$$(k+1)(3f(k+1)+3)n_{k+1} - 2((k+1)n_{k+1} + 1)$$

$$> (k+1)(3f(k+1)+1)n_{k+1} > (k+1)n_{k+1} + 1$$

Hence we can find $(k+1)n_{k+1} + 1$-many elements in $A_{k+1} \cap P_{k+1}$ above $\sum_{i=1}^{k+1} in_i + 1$. Also, since $|A \cap P_n| > f(k+1)$ we can make sure that $a_{\sum_{i=1}^{k+1} in_i+1} > f(k+1)$. This way we ensure that:

(1) Since $a_{a'_{k-1}}$ was not previously defined in previous rounds, $a'_k := a_{a'_{k-1}} > \sum_{i=1}^{k+1} in_i + 1$ and $a_{a'_k}$ has not been defined yet. Hence a future $a'_{k+1} := a_{a'_k} > f(k+1)$.
(2) $|B \cap P_{n_{k+1}}| > (k+1)n_{k+1} + 1$.

Set $B = \{a_n \mid n < \omega\}$. So by the construction, for every k there is n_k such that $|B \cap P_{n_k}| > kn_k$. Hence $B \in I^+$. Also, note that $f_B(n) = a_n$ since the a_n's are increasing. By the construction and definition of $exp(f)$, $exp(f_B)(n) = a'_n > f(n)$. Finally, note that for each n, there is k such that for every $k' \geq k$, $a_{k'} \in A_m$ for some $m \geq n$. Since the sequence of A_n's is $\subseteq$-decreasing, $a_{k'} \in A_n$. We conclude that $B \setminus A_n \subseteq \{a_0, ..., a_k\}$. □

Now for the construction of the ultrafilter. Enumerate $P(\omega) = \langle X_\alpha \mid \alpha < \omega_1\rangle$, and $P(\omega)^\omega = \langle \vec{A}_\alpha \mid \alpha < \omega_1\rangle$ such that each sequence in $P(\omega)^\omega$ appears cofinaly many times in the enumeration. Also enumerate $\omega^\omega = \langle \tau_\alpha \mid \alpha < \omega_1\rangle$. We define a sequence of filters V_α such that:

(1) $\beta < \alpha \Rightarrow V_\beta \subseteq V_\alpha$.
(2) $V_\alpha \subseteq I^+$.
(3) Either X_α or $\omega \setminus X_\alpha \in V_{\alpha+1}$.
(4) There is $X \in V_{\alpha+1}$ such that $\tau_\alpha < exp(f_X)$.
(5) If $\vec{A}_\alpha \subseteq V_\alpha$ then there is a pseudo-intersection $A \in V_{\alpha+1}$.

Let $V_0 = I^*$. At limit steps δ we define $V_\delta = \bigcup_{\beta<\delta} V_\beta$. It is clear that $(1) - (5)$ still holds at limit steps. At successors, given V_α, since we have only performed countably many steps so far, there are sets $B_n \in V_\alpha$ such that $V_\alpha = I^*[\langle B_n \mid n < \omega\rangle]$ where B_n is $\supseteq$-decreasing. If either X_α or $\omega \setminus X_\alpha$ is already in V_α, we ignore it. Otherwise, we must also have $V_\alpha[X_\alpha] \subseteq I^+$. If $\vec{A}_\alpha \not\subseteq V_\alpha[X_\alpha]$ ignore it. Otherwise, enumerate the set

$$\{B_n \cap X_\alpha \mid n < \omega\} \cup \{\vec{A}_\alpha(n) \mid n < \omega\} \subseteq V_\alpha[X_\alpha]$$

by $\langle B'_n \mid n < \omega\rangle$ and let $C_n = \cap_{m \leq n} B'_m$. We apply the previous lemma to the sequence $\langle C_n \mid n < \omega\rangle$, and τ_α to find $A^* \subseteq \omega$ such that:

(1) $A^* \in I^+$.
(2) $exp(f_{A^*}) > \tau_\alpha$.
(3) $A^* \subseteq^* C_n$ for every n.

Since for every $n < \omega$, there is n' such that $C_{n'} \subseteq \vec{A}_\alpha(n) \cap B_n$, $A^* \subseteq^* \vec{A}_\alpha(n)$, namely A^* is a pseudo intersection of both $\langle B_n \mid n < \omega\rangle$ and $\vec{A}_\alpha$. Also, A^* is a positive set with respect to the ideal $V_\alpha[X_\alpha]$. Otherwise, there is some $A \in I^*$ and B_n such that $A^* \cap (A \cap B_n \cap X_\alpha) = \emptyset$. But then $(A^* \cap B_n \cap X_\alpha) \cap A = \emptyset$ which implies that $A^* \cap B_n \cap X_\alpha \in I$. However, $A^* \subseteq^* B_n \cap X_\alpha$, which implies that $A^* \in I$, contradicting property (1) above in the choice of A^*. Hence we can define $V_{\alpha+1} = V_\alpha[X_\alpha, A^*]$ and $(1) - (5)$ hold.

This concludes the recursive definition. The ultrafilter witnessing the theorem is defined by $V^* = \bigcup_{\alpha<\omega_1} V_\alpha$.

Proposition 4.18. *V^* is a non-rapid almost-rapid p-point ultrafilter.*

Proof. V^* is an ultrafilter since for every $X \subseteq \omega$, there is α such that $X = X_\alpha$ and so either X_α or $\omega \setminus X_\alpha$ are in $V_{\alpha+1} \subseteq V^*$. Also V^* is a p-point since if $\langle A_n \mid n < \omega\rangle \subseteq V^*$ then there is $\alpha < \omega_1$ such that $\langle A_n \mid n < \omega\rangle \subseteq V_\alpha$ and by the properties of the enumeration there is $\beta > \alpha$ such that $\vec{A}_\beta = \langle A_n \mid n < \omega\rangle$. This means that in $V_{\beta+1}$ there is a pseudo intersection for the A_n's. It is non-rapid as $V^* \subseteq I^+$ and, in fact, the sequence P_n witnesses that it is non-rapid. Finally, it is almost rapid since for any function $\tau : \omega \to \omega$, there is α such that $\tau = \tau_\alpha$ and therefore in $V_{\alpha+1}$ there is a set X such that $exp(f_X) > \tau$. □

□

Corollary 4.19. *It is consistent that there is a p-point which is not rapid but still above ω^ω.*

Remark 4.20. CH is not necessary in order to obtain such an ultrafilter, since we can, for example, repeat a similar argument in the iteration of Mathias forcing after we forced the failure of CH and obtain such an ultrafilter. In fact, we conjecture that the construction of Ketonen [22] of a p-point from $\mathfrak{d} = \mathfrak{c}$ can be modified to get a non-rapid almost-rapid p-point.

5. QUESTIONS

We collect here some problems which relate to the work of this paper. The first batch of questions regards the Tukey-type of sums of ultrafilters:

Question 5.1. If $\mathbb{P}$ is below $\mathcal{B}(U, V_\alpha)$ does it imply that $\mathbb{P}$ is uniformly below $\mathcal{B}(U, V_\alpha)$? What about the case where $\mathbb{P}$ is an ultrafilter?

Question 5.2. Is it true in general that $\sum_U V_\alpha = \inf \mathcal{B}(U, V_\alpha)$?

Question 5.3. Is there a nice characterization for the Tukey-type of $\sum_U V_\alpha$ if we assume that $V_0 \geq_T V_1 \geq_T V_2 ....$?

Question 5.4. Is there a nice characterization for the Tukey-type of $\sum_U V_\alpha$ when the sequence of V_α's is discrete?

How much of the theory developed here generalizes to measurable cardinals? more concretely:

Question 5.5. Does Lemma 1.18 hold true for σ-complete ultrafilters over uncountable cardinals?

The next type of questions relate to the I-pseudo intersection property

Question 5.6. What is the characterization of the I-p.i.p property in terms of Skies and Constellations of ultrapowers from [30]?

Question 5.7. Is the equivalence of Proposition 4.2 true for every ideal I?

The following addresses the commutativity of cofinal types between ultrafilters on different cardinals.

Question 5.8. Is it true that for every two ultrafilters U, V on any cardinals, $U \cdot V \equiv_T V \cdot U$?

Let us note that if U, V are λ and κ ultrafilters respectively, then this holds. To see this, first note that if $\kappa = \lambda \geq_T \omega$, then this is a combination of the results from [2] and this paper's main result. Without loss of generality, $\lambda < \kappa$. In which case, $U \cdot V = U \times V^\lambda$. Since V is λ^+-complete, it is not had to see that $V^\lambda \equiv_T V$, hence $U \cdot V \equiv_T V \cdot U$. On the other hand, since $2^\lambda < \kappa$, every set $X \in V \cdot U$, contained a set of the form $X \times Y$ for $X \in V$ and $Y \in U$. It follows that also $V \cdot U \equiv_T U \times V \equiv_T U \cdot V$. So the question above is interesting in case we drop the completeness assumption on the ultrafilters.

the results in § 4 suggest that the Tukey-type of I^ω as an important role in the analysis of non Tukey-top ultrafilters. Note that the same argument which worked for ω^ω in Proposition 4.1 works for I^ω, hence leading to the conclusion that if I is an analytic ideal the no non-principal ultrafilter can be Tukey equivalent to I^ω. The following question is more open-ended:

Question 5.9. How can we force different values of the Tukey-type of I^ω, when $I \not\equiv_T \omega^\omega$?

Finally, we present a few questions regarding the new class of α-almost-rapid ultrafilters.

Question 5.10. Is it true that for every $\alpha < \beta < \omega_1$, the class of α-almost-rapid ultrafilters is consistently strictly included in the class of β-almost-rapid ultrafilters?

We conjecture a positive answer to this question and that similar methods to the one presented in Theorem 4.16 under CH should work.

Question 5.11. Does $\mathfrak{d} = \mathfrak{c}$ imply that there is a p-point which is almost-rapid but not rapid?

Ultimately we would like to understand if such ultrafilters exist in ZFC:

Question 5.12. Is it consistent that there are no almost-rapid ultrafilters?

Following Miller, a natural model would be adding $\aleph_2$-many Laver reals. However, the current argument does not immidietly rule out α-almost rapid ultrafilters.

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