MATH 504: CHAPTER 3 FORCING

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1. History of forcing

Paul Cohen invented the method of forcing to construct models of ZFC in a collection of works which awarded hi, with the fields medal and its influence set theory and the foundations of mathematics till nowdays. Since Cohen's original usage of Forcing this method has been sophisticated and has been used for many other purposes, but the core idea stays the same. The way we should think of forcing is as follows:

While inner models provide a way of constructing models $M \subseteq V$, forcing can be thought of as providing models $V \subseteq M$. This idea can be made precise using reflection theorems which we will explain later.

The idea is to start with V, which we call "the ground model" over which we "force" (namely extend the model using the method of forcing) and extend V to a model M by adding some new object which does not appear in V, together with everything that has to be added in order to satisfy ZFC. This idea is quite similar to field extension, where given a filed K and some polynomial p with no root in K, we can add to K a root of α of p and let $K[\alpha]$ be the minimal field extending $K \cup \{\alpha\}$. In the field extension $K[\alpha]$, every object is the result of plugging α into some polynomial $q(x)$, and we can think of $q(x)$ as "names" or "expressions" for elements in $K[\alpha]$.

In forcing we will add this object we call "V-generic filter" $G \notin V$ and extend V to the generic extension $V[G]$. Similarly, we will have "names" for each object $x \in V[G]$. The analog of the polynomial p will be what we call a "forcing notion" which is simply some poset $\mathbb{P}, \leq_{\mathbb{P}}$ with a minimal element $1_{\mathbb{P}}$. This will enable us to talk about objects in $V[G]$ while working in V.

The metamathematical justification for the method of forcing is as follows: Suppose that we have used the method of forcing to produce a model $V \subseteq M$ of $ZFC+\neg CH$. We want to prove that $Con(ZFC+\neg CH)$. If $\neg Con(ZFC+\neg CH)$, then there are finitely many axioms in $ZFC + \neg CH$ which proves some contradiction. We use reflection to find a countable transitive model M of ZFC which satisfies the finitely many axioms which appear in ZFC (formally, we should also add finitely many axioms that suffice to carry the argument of forcing). As we will see, we can always find those M-generic filters in V (given that M is countable). Since the model obtained by forcing satisy $\neg CH$, we have produced a set model $M \subseteq N$ which satisfy finitely

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many axioms which proves a contradiction, so $N \models \psi \wedge \neg\psi$ and $\psi^N \wedge \neg\psi^N$ holds in V, a contradiction.

2. Forcing notions

Definition 2.1. A forcing notion is a poset $\langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ which has a greatest element $1_{\mathbb{P}}$.

Example 2.2. Cohen forcing

$$Add(\omega,1) = \{f : \omega \to 2 \mid |f| < \omega\},\ \ f \leq g \text{ iff } g \subseteq f.$$

Here $1 = \emptyset$.

Example 2.3. Levy collapse $Col(\omega,\omega_1) = \{f : \omega \to \omega_1 \mid |f| < \omega\}$ ordered as above.

Example 2.4. Mathias forcing

$$M_\omega := \{(s,A) \in [\omega]^{<\omega} \times A \in [\omega]^\omega \mid \min(A) > \max(s)\}$$

The order is $(s,A) \geq (t,B)$ iff $t \cap \{0,...,\max(s)\} = s$, $t \setminus s \subseteq A$ and $B \subseteq A$.

Example 2.5. Random Real forcing $R = \{B \subseteq [0,1] \mid \mu(B) > 0 \wedge B \text{ is a borel set}\}$ the order is $B \leq C$ iff $B \subseteq C$.

Example 2.6. Shooting a club through a stationary set $S \subseteq \omega_1$

$$\mathbb{P}(S) := \{c \subseteq S \mid c \text{ is closed bounded}\}$$

the order here is an end extension.

Definition 2.7. A subset $D \subseteq \mathbb{P}$ is said to be dense, if:

$$\forall p \in \mathbb{P}\ \exists q \leq p\ q \in D$$

Example 2.8. For every n,

$$D_n = \{f \in Add(\omega,1) \mid n \in \text{dom}(f)\}$$

is dense, since if $f \in Add(\omega,1)$, either $n \in \text{dom}(f)$ and then f is already in D_n or $n \notin \text{dom}(f)$ and we can extend $f \geq g = f \cup \{\langle n,0\rangle\}$.

Given $h : \omega \to \omega$, the set

$$D_h = \{f \mid \exists n \in \text{dom}(f)\ f(n) \neq h(n)\}$$

is dense.

Example 2.9. For every $\delta < \omega_1$, the set

$$D_\delta := \{g \in Col(\omega,\omega_1) \mid \delta \in Im(g)\}$$

is dense.

Example 2.10. Given any A such that $\mu(A) = 1$, the set

$$D_A = \{B \in R \mid B \subseteq A\}$$

is dense. Indeed, let $B \in R$ then $A \cap B$ is measurable and $\mu(A \cap B) > 0$ (since B is positive and A is measure 1). By regularity of the Lebesgue measure, there is a closed (hence Borel) set $C \subseteq A \cap B$ such that $\mu(C) > 0$.

Definition 2.11. A subset $G \subseteq \mathbb{P}$ is a filter if:

(1) $\forall p \in \mathbb{G}$ and $q \geq p$, we have that $q \in G$.
(2) For every $p, q \in G$ there is $r \in G$ such that $p, q \geq r$.
(3) $G \neq \emptyset$

G is called V-generic if for every dense subset of $\mathbb{P}$, $G \cap D \neq \emptyset$.

Example 2.12.

Theorem 2.13. *If M is countable then for every $\mathbb{P} \in M$, there for every $q \in \mathbb{P} \cap M$, there is an M-generic filter for $G \subseteq \mathbb{P}$ with $q \in G$*

Theorem 2.14. *Suppose that $\mathbb{P}$ is splitting (namely $\forall p \in \mathbb{P}$ there are $q, r \leq p$ such that q, r are incompatible). Then if G is an M-generic filter for $\mathbb{P}$, $G \notin M$.*

3. The generic extension

We are now ready to construct the generic extension $M[G]$.

Definition 3.1. Let $\mathbb{P} \in M$ be a forcing notion. We define in M by $\in$-induction what a $\mathbb{P}$-name is. $\emptyset$ is a $\mathbb{P}$-name. Suppose that I is a set and for each $i \in I$, $\dot{\tau}_i$ is a $\mathbb{P}$-name and $p_i \in \mathbb{P}$ the also

$$\{\langle p_i, \dot{\tau}_i \rangle \mid i \in I\}$$

is a $\mathbb{P}$-name

We denote the class of all $\mathbb{P}$-names by $V^{\mathbb{P}}$ (in particular there is a formula with a parameter $\mathbb{P}$ which asserts that x is a $\mathbb{P}$-name)

Definition 3.2. Let $\mathbb{P} \in M$ be a forcing notion and $G \subseteq \mathbb{P}$ be M-generic. The interpretation of a $\mathbb{P}$-name $\dot{\tau} = \{\langle p_i, \dot{\tau}_i \rangle \mid i \in I\}$ under the generic filter G is defined recursively by:

$$(\emptyset)_G = \emptyset$$
$$(\dot{\tau})_G = \{(\dot{\tau}_i)_G \mid i \in I, p_i \in G\}$$

Definition 3.3. The generic extension of M by G is defined by:

$$M[G] := \{(\dot{\tau})_G \mid \dot{\tau} \in V^{\mathbb{P}}\}$$

Our goal is to prove the following

Theorem 3.4. *Let M be a transitive model so f ZFC and $\mathbb{P} \in M$ be any forcing notion and G be an M-generic filter. Then:*

(1) $M \subseteq M[G]$.
(2) $M[G] \models ZFC$.
(3) $G \in M$.
(4) $On^M = On^{M[G]}$.
(5) *If N is a model of ZFC such that $M \subseteq N$, $G \in N$ then $M[G] \subseteq N$.*

To see that $M \subseteq M[G]$, we have the following definition

Definition 3.5. A canonical name for an element $x \in M$ is defined recursively by

$$\hat{X} := \{\langle \hat{x}, 1_{\mathbb{P}} \rangle \mid x \in X\}$$

Proposition 3.6. *$(\hat{X})_G = X$ for every $X \in M$ and therefore $M \subseteq M[G]$.*

Proof. By induction, and since $1_{\mathbb{P}} \in G$,

$$(\hat{X})_G = \{(\hat{x})_G \mid x \in X\} = \{x \mid x \in X\} = X$$

$\square$