Commutativity of cofinal types of ultrafilters

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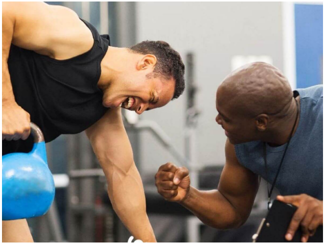
Motivation



Let $(X, \tau_X), (Y, \tau_Y)$ be Hausdorff topological spaces. Recall that:

Definition 1

A function $f : X \to Y$ is continuous in the sequential sense if whenever $(x_n)_{n=0}^{\infty} \subseteq X$ is a sequence converging to $x \in X$ (namely, for every neighborhood $U \in \mathcal{N}(x)$ there is N such that for all $n \geq N$, $x_n \in U$), the sequence $(f(x_n))_{n=0}^{\infty}$ converges to $f(x)$.

$\Rightarrow$ Sequential continuity is equivalent to continuity in spaces where the following holds: $x \in cl(Z)$ iff there is a sequence $(z_n)_{n=0}^{\infty} \subseteq Z$ which converges to x.

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- ⇒ Sequential continuity is equivalent to continuity in spaces where the following holds: $x \in cl(Z)$ iff there is a sequence $(z_n)_{n=0}^{\infty} \subseteq Z$ which converges to x.
- ⇒ For example in first-countable spaces a function f is continuous if and only if f is continuous in the sequential sense.
- ⇒ The two are not equivalent: for example $f : \omega_1 + 1 \to \mathbb{R}$ defined by $f(x) = 0$ if $x < \omega_1$ and $f(\omega_1) = 1$ is not continuous but sequentially continuous.)

Definition 2 (Moore-Smith 1922)

Let $(A, \leq_A)$ be a directed set. An *A-net* is a function $\vec{x} = (x_a)_{a \in A}$. A point x is a limit of $\vec{x}$ if for every $U \in \mathcal{N}(x)$ there is a such that , $b \geq a$, $x_b \in U$ (a.k.a Moore-Smith convergence).

$\Rightarrow$ A function $f : X \to Y$ is continuous iff for every net $(x_a)_{a \in A}$ with limit x, $f(x)$ is a limit of $(f(x_a))_{a \in A}$.

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- ⇒ A function $f : X \to Y$ is continuous iff for every net $(x_a)_{a \in A}$ with limit x, $f(x)$ is a limit of $(f(x_a))_{a \in A}$.
- ⇒ $x \in cl(Z)$ iff there is a net $\vec{x} \subseteq Z$ converging to x. For example, one might take $(z_U)_{U \in \mathcal{N}(x)}$ where $z_U \in U \cap Z$.

Cofinal maps

Some "types" of directed sets actually give essentially the same notion of net, for example, $\mathbb{N}$ and $\mathbb{N}_{even}$ or even $fin = \{X \in P(\mathbb{N}) \mid X \text{ is finite}\}$. More generally we would like to find an equivalence relation that reduces to the "essential" ordered sets. This is given by the Tukey order which was defined by J. Tukey:

Definition 3 (Tukey '40 [12])

Let $(P, \leq_P), (Q, \leq_Q)$ be two partially ordered (directed) sets. Define $(P, \leq_P) \leq_T (Q, \leq_Q)$ iff there is a cofinal map[a] $f : Q \to P$. Define $(P, \leq_P) \equiv_T (Q, \leq_Q)$ iff $(P, \leq_P) \leq_T (Q, \leq_Q)$ and $(Q, \leq_Q) \leq_T (P, \leq_P)$.

[a] if for every cofinal $B \subseteq Q$, $f[B] \subseteq P$ is cofinal.

If $B \leq_T A$, then any B-net $(x_b)_{b \in B}$ can be now replaced by $(x_{f(a)})_{a \in A}$ and if x is a limit point of $(x_b)_{b \in B}$ then x must be a limit of $(x_{f(a)})_{a \in A}$.
The research of what are the "essential" A's is a completely set theoretic (order theoretic) question.

Classic results of Todorcevic

Theorem 4 (Todorcevic '85 [11])

It is consistent that there are exactly 5 *Tukey classes of directed posets of cardinality at most* $\aleph_1$.

Theorem 5 (Todorcevic '85 [11])

for any regular $\kappa > \omega$, *there are* 2^κ*-many distinct Tukey classes of cardinality* $\kappa^{\aleph_0}$. *In particular, if* $\mathfrak{c}$ *is regular, then there are at least* $2^{\mathfrak{c}}$ *many distinct Tukey classes of cardinality* $\mathfrak{c}$.

Definition 6

Given a net $\vec{x} = (x_a)_{a \in A}$, define for each $a \in A$, $x_{\geq a} = \{x_b \mid b \geq a\}$. The filter associated with $\vec{x}$, denoted by $F_{\vec{x}}$ is the filter generated by the sets $x_{\geq a}$. Namely, $T \in F_{\vec{x}}$ iff $\exists a \in A,\ x_{\geq_a} \subseteq T$.

The filter $F_{\vec{x}}$ determines the convergence properties of the net $\vec{x}$ in the sense that $\vec{x}$ converges to x iff $\mathcal{N}(x) \subseteq F_{\vec{x}}$. This gives rise to the idea of converging filters:

Definition 7 (H. Cartan '37)

We say that a filter F converges to a point x if $\mathcal{N}(x) \subseteq F$.

Since every filter can be extended to an ultrafilter, if F converges to a point x then there is an ultrafilter which covergese to x as well. Therefore, for most purposes, it suffices to consider only ultrafilters, or *ultranets*. For example, TFRE:

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- $f : X \to Y$ is continuous.
- For every $x \in X$, and every ultrafilter U such that $\mathcal{N}(x) \subseteq U$, the ultrafilter $f_*(U) = \{B \subseteq Y \mid f^{-1}[B] \in U\}$ extends $\mathcal{N}(f(x))$.

The Tukey order on ultrafilters

All of the above motivates the study of cofinal types of ultrafilters from a topological point of view, and more precisely, the directed order $(U, \supseteq)$ where U is an ultrafilter. On ω, this has been studied extensively by Blass, Dobrinen, Milovich, Raghavan, Shelah, Solecki, Todorcevic and many others.

Proposition 1

Suppose that $U \leq_T V$ where U, V are ultrafilters, then there is a (weakly) monotone map $f : V \to U$ such that $Im(f)$ is cofinal in U.

- $\Rightarrow$ It is clearly the functions to compare the minimal size of a base (and therefore to understand the ultrafilter number).
- $\Rightarrow$ $U \leq_{RK} V$ implies $U \leq_T V$.

Entering the realm of large cardinals

(joint with Natasha Dobrinen)

The Tukey-top class

An ultrafilter U in ω is called Tukey-top if for every ultrafilter W on ω, $W \leq_T U$.

Theorem 8 (Isbell '65 [8])

There exists a Tukey top ultrafilter.

Question (Isbell '65)

Are there provably a non-Tukey-top ultrafilters on ω?

Theorem 9 (B.-Dobrinen '23[3])

Let U be a κ-complete ultrafilter over κ, then U is Tukey-top (wrt. κ-complete ultrafilters) iff $\neg Gal(U, \kappa, 2^\kappa)$, that is: there is a sequence $\langle X_i \mid i < 2^\kappa \rangle \subseteq U$ such that for every $I \in [2^\kappa]^\kappa$, $\bigcap_{i \in I} X_i \notin U$.

Generalizing Isbell's construction, we proved the following:

Theorem 10 (B.-Dobrinen '23)

If κ is κ-compact, there is a κ-complete ultrafilter over κ which is Tukey-top. (forcing constructions B.-Garti-Shelah [4] B.-Gitik [5])

Recall that U is a p-point on $\kappa \geq \omega$ if for every sequence $\langle X_\alpha \mid \alpha < \kappa \rangle \subseteq U$ there is $X \in U$ which is a pseudo intersection for the sequence i.e. for every $\alpha < \kappa$, $X \setminus X_n$ is bounded (or finite if $\kappa = \omega$).

Theorem 11 (B. '22[1])

If U is an n-fold sum of p-points the U is not Tukey-top. Hence if in $L[\mathbb{E}]$ there is no measurable limit of superstrong cardinals, then there are no κ-complete Tukey-top ultrafilters over κ.

Theorem 12 (B.-Goldberg '23 [6])

Assume UA and that every irreducible is Dodd-sound. Then the following are equivalent for every κ-complete ultrafiler U over κ:

1. *U is Tukey-top (i.e. $\neg Gal(U, \kappa, 2^\kappa)$)*
2. *U is not an n-fold sum of p-points.*
3. *$\diamondsuit^*_{thin}(U)$.*

The result above is also true for σ-complete ultrafilters over regular cardinals.

Fubini products

Fact 13

Let $(P, \leq_P), (Q, \leq_Q)$ be directed orders. Then[a] *$(P \times Q, \leq_\times)$ is the least upper bound of P, Q in the Tukey order. Hence $P \equiv_T P \times P$.*

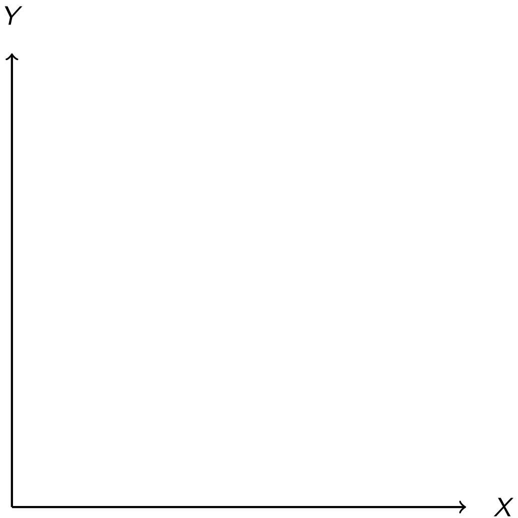
[a] $(p, q) \leq_\times (p', q')$ if and only if $p \leq_P p'$ and $q \leq_Q q'$.

Definition 14 (Fubini product)

Suppose that U is an ultrafilter over X and V an ultrafilter over Y. We denote by $U \cdot V$ the Fubini product of U and V which is the ultrafilter defined over $X \times Y$ as follows, for $A \subseteq X \times Y$,

$$A \in U \cdot V \text{ if and only if } \{x \in X \mid (A)_x \in V\} \in U$$

where $(A)_x = \{y \in Y_x \mid \langle x, y \rangle \in A\}$. If $U = V$, then U^2 is defined as $U \cdot U$ and referred to as the Fubini power.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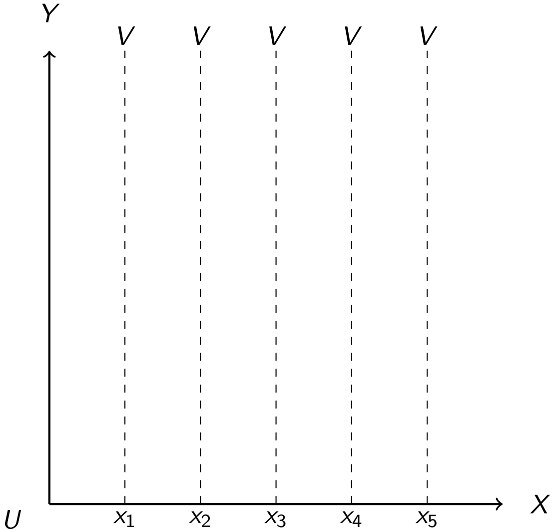
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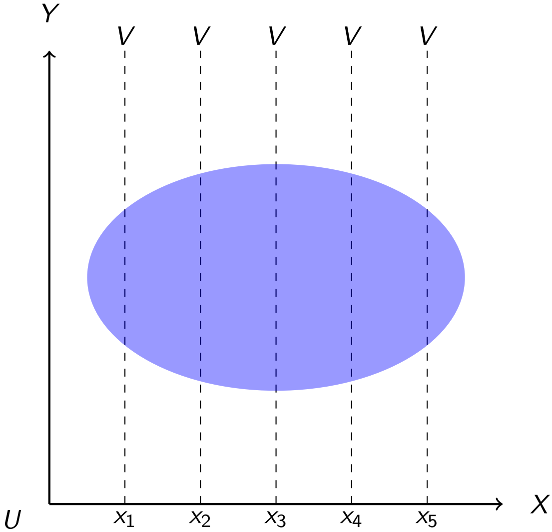
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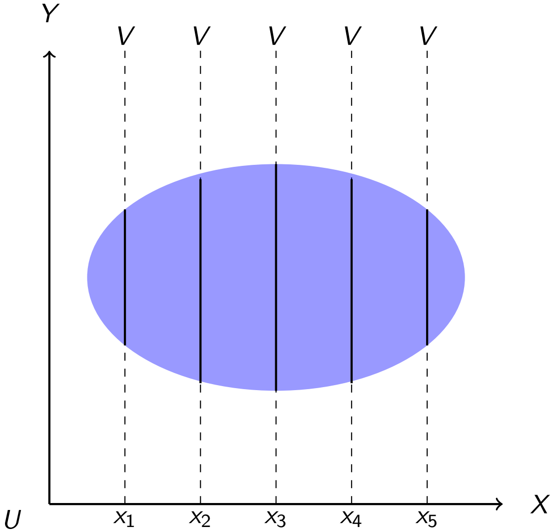

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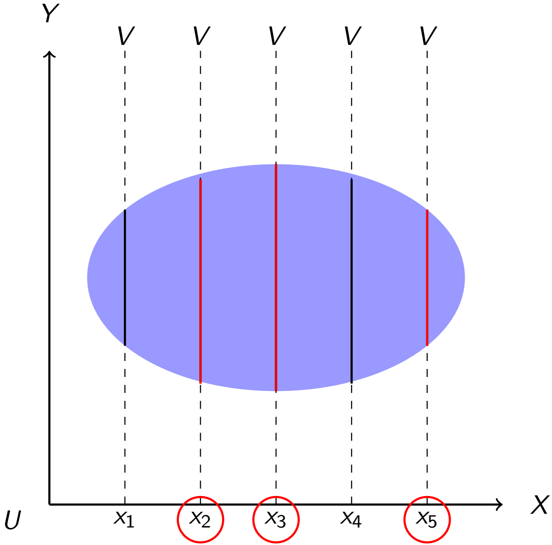
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$(U,\supseteq),(V,\supseteq)\leq_T (U\cdot V,\supseteq)$. Therefore $(U\times V,\leq_\times)\leq_T (U\cdot V,\supseteq)$.

Theorem 15 (Dobrinen-Todorcevic-Milovich '12 [7, 9])

For any two ultrafilters U,V on ω, $U\cdot V\equiv_T U\times\prod_{n<\omega}V$. The order on Cartesian products is always coordinatewise. In particular $V\cdot V\equiv_T\prod_{n<\omega}V$ and $U\cdot V\equiv_T U\cdot(V\cdot V)$

These results also hold for to κ-complete ultrafilters over κ.

Theorem 16 (B.-Dobrinen '23)

Let U,V be any κ-complete ultrafilters over $\kappa>\omega$, then $U\cdot V\equiv_T U\times V$. In particular $U\cdot V\equiv_T V\cdot U$ and $U\cdot U\equiv_T U$.

The proof essentially uses the well-ordering of κ^κ/U which is a virtue of the σ-completeness.

Back to earth (ω)

Commutativity of Cofinal types on ω

Definition 17

As ultrafilter U over ω is rapid if for every function $f : \mathbb{N} \to \mathbb{N}$ there is $X \in U$ such that for every $n < \omega$, $X(n) \geq f(n)$.

Theorem 18 (Dobrinen-Todorcevic '11)

Suppose that V, U are ultrafilters on ω, V is a rapid p-point. Then $U \cdot V \equiv_T U \times V$. In particular, if U, V are rapid p-points then $U \cdot V \equiv_T V \cdot U$.

In particular if U is a rapid p-point then $U \cdot U \equiv_T U$. Dobrinen and Todorcevic constructed an example of a p-point U such that $U <_T U^2$.

Theorem 19 (Milovich '12)

If U, V are ultrafilters on ω and U is a p, then $V \cdot U \equiv_T V \times U \times \omega^\omega$ and therefore if U, V are both p-points then $U \cdot V \equiv_T V \cdot U$.

Theorem 20 (B. '24 [2])

For any two ultrafilters U, V on ω, $U \cdot V \equiv_T V \cdot U$.

The proof

For a filter F we denote by $F^* = \{A^c \mid A \in F\}$ the dual ideal of F. Also, denote by $fin = \{A \subseteq \omega \mid A \text{ finite}\}$. Note that $(F, \supseteq) \simeq (F^*, \subseteq)$.

Definition 21

Suppose that U is an ultrafilter and $I \subseteq U^*$ is an ideal. We say that U has the I-p.i.p if for any sequence $\langle X_n \mid n < \omega\rangle \subseteq U$, there is $X \in U$ such that for every $n < \omega$, $X \setminus X_n \in I$.

For example, U is a p-point if and only if U has the fin-p.i.p.

Proposition 2

Suppose that U has the I-p.i.p, then $U \cdot U \equiv_T \prod_{n<\omega} U \leq_T U \times \prod_{n<\omega} I$.

Since $fin \equiv_T \omega$, we get that for p-points U, $U \cdot U \leq_T U \times \omega^\omega$ (this fact about p-points was already known to Dobrinen and Todorcevic).

Theorem 22

If U and V are ultrafilters, then U (and V of course) have the $(U \cap V)^*$-p.i.p.

Corollary 23

$U \cdot V \leq_T U \times V \times \prod_{n<\omega} U \cap V$.

Theorem 24

$U \cdot V \equiv_T U \times V \times \prod_{n<\omega} U \cap V$.

To prove the theorem it remains to prove that $U \cdot V \geq_T U \times V \times \prod_{n<\omega} U \cap V$, and by the least upper bound property, it remains to prove that $U \cdot V \geq_T \prod_{n<\omega} U \cap V$.

Lemma 25 (Proof omitted)

For every $F \subseteq V$, $V \cdot V \geq_T F$.

By the lemma, we conclude that $V \cdot V \geq_T U \cap V$ and

$$U \cdot V \equiv_T U \cdot (V \cdot V) \equiv U \times \prod_{n<\omega} (V \cdot V) \geq_T \prod_{n<\omega} U \cap V$$

Corollary 26

For every ultrafilters U, V, $U \cdot V \equiv_T V \cdot U$.

Diamond-like Principles on ω

Definition 27 (Jensen)

We say that

1. $\diamondsuit(\kappa)$ holds if there is a sequence $\langle A_\alpha \mid \alpha < \kappa\rangle$, $A_\alpha \subseteq \alpha$ such that for **every** set $X \subseteq \kappa$, $\{\alpha < \kappa \mid X \cap \alpha = A_\alpha\}$ is stationary.
2. $\diamondsuit^-(\kappa)$ is there is a sequence $\langle \mathcal{A}_\alpha \mid \alpha < \kappa\rangle$ such that for every $\alpha < \kappa$, $\mathcal{A}_\alpha \in [P(\alpha)]^{\leq|\alpha|}$ such that for every $X \subseteq \kappa$, $\{\alpha < \kappa \mid X \cap \alpha \in \mathcal{A}_\alpha\}$ is stationary.
3. $\diamondsuit^*(\kappa)$ holds if there is a sequence $\langle \mathcal{A}_\alpha \mid \alpha < \kappa\rangle$, such that $\mathcal{A}_\alpha \in [P(\alpha)]^{\leq|\alpha|}$ such that for every $X \subseteq \kappa$, $\{\alpha < \kappa \mid X \cap \alpha \in \mathcal{A}_\alpha\}$ contains a club.

Of course, non of these makes sense on ω.

Idea

Replace the club filter with a general filter F

Definition 28

Let F be a filter over a cardinal $\kappa \geq \omega$, and let $\pi : \kappa \to \kappa$ we say that:

1. $\diamondsuit^*_\pi(F)$, holds if there is a sequence $\langle \mathcal{A}_\alpha \mid \alpha < \kappa \rangle$, $\mathcal{A}_\alpha \in [P(\alpha)]^{\leq \pi(\alpha)}$. such that for every set X, $\{\alpha < \kappa \mid X \cap \alpha \in \mathcal{A}_\alpha\} \in F$.
2. $\diamondsuit^-_\pi(F)$, holds if there is a sequence $\langle \mathcal{A}_\alpha \mid \alpha < \kappa \rangle$, $\mathcal{A}_\alpha \in [P(\alpha)]^{\leq \pi(\alpha)}$. such that for every set X, $\{\alpha < \kappa \mid X \cap \alpha \in \mathcal{A}_\alpha\} \in F^+$.

1. If F is an ultrafilter then the above definition coincide
2. If $\pi(\alpha) = 1$ then we $\diamondsuit^-_\pi(cub_\kappa)$ is just the usual diamond.
3. Trivial if we allow $\pi(\alpha) \approx 2^\alpha$
4. If U is a Dodd-sound ultrafilter then $\diamondsuit^-_\pi(U)$, where $\pi(\alpha) = 2^{\tau(\alpha)}$, $[\tau]_U = \kappa$.

Theorem 29 (B.-Wu)

If $\sum_{n=0}^{\infty} \frac{\pi(n)}{2^n} < \infty$*, and* F *extends the Frechet filter then* $\diamondsuit_{\pi}^{-}(F)$ *fails.*

Proof.

Denote by $\mathbb{P}$ the standard Borel measure on 2^ω (identified with $P(\omega)$). Suppose that $\langle \mathcal{A}_n \mid n < \omega \rangle$ witness that $\diamondsuit_{\pi}^{-}(F)$ holds, and consider the events

$$E_n = \{X \in P(\omega) \mid X \cap n \in \mathcal{A}_n\}$$

Then $\mathbb{P}(E_n) = \frac{\pi(n)}{2^n}$. By the Borel-Cantelli lemma, if $\sum_{n=0}^{\infty} \mathbb{P}(E_n) < \omega$ then $\mathbb{P}(\limsup E_n) = 0$, where $\limsup E_n = \bigcap_{n<\omega} \bigcup_{m \geq n} E_n$. Therefore there is $X \notin \limsup E_n$, but then $\{n < \omega \mid X \in E_n\} \in F^+$ is finite, so F cannot extend the Frechet filter. □

Corrected definition:

Definition 30

$\diamondsuit^-_\pi(U)$ assures the existence of a sequence $\langle \mathcal{A}_\alpha \mid \alpha < \kappa \rangle$ such that $\mathcal{A}_\alpha \in [P(\alpha)]^{\leq \pi(\alpha)}$ such that there are 2^κ-many sets $X \subseteq \kappa$ such that $\{\alpha < \kappa \mid X \cap \alpha \in \mathcal{A}_\alpha\} \in U$.

Theorem 31 (B.-Goldberg)

Let U be an ultrafilter, and suppose that π is not almost one-to-one modulo U. If $\diamondsuit^-_\pi(U)$ holds then U is Tukey-top.

The proof generalizes to ultrafilters on ω as well.

Theorem 32 (B.-Wu)

Let π be any infinite-to-one function. It is ZFC provable that there is an ultrafilter U such that π is not almost one-to-one modulo U and $\diamondsuit^-_\pi(U)$ hold.

Thank you for your attention!

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Theorem 33 (Dobrinen-Todorcevic)

For p-point ultrafilters the following are equivalent:

1. $U \equiv_T U \cdot U$.

Dobrinen and Todorcevic forced p-point which is not above ω^ω and therefore cannot be Tukey equivalent to its Fubini square.
By results of Solecki and Todorcevic [10], an ultrafilter U cannot be Tukey equivalent to ω^ω.

Question (Dobrinen)

Is being Tukey above ω^ω equivalent to being rapid?

Question (Dobrinen-Todorcevic)

Is there always an ultrafilter which is not Tukey above ω^ω?

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Theorem 34 (B. '24)

If U is not Tukey above ω^ω then U must be a p-point. In particular, in Shelah's model where there are no p-points every ultrafilter is Tukey above ω^ω.

Theorem 35 (B. '24)

Assume CH. Then there is a p-point ultrafilter U over ω such that $U \geq_T \omega^\omega$ but U is not rapid.

Definition 36

U is called almost rapid if for any function $f : \omega \to \omega$, there is $X \in U$, such that f_X dominates f, where f_X is defined recursively, $f_X(0) = \min(X)$, $f_X(n+1)$ is the $f_X(n)^{\text{th}}$ element of X

Now it is not hard to see that $X \mapsto f_X$ is a monotone map and if U is almost rapid, this map is cofinal. Hence if U is almost rapid, then $U \geq_T \omega^\omega$. Under CH, I proved that we can construct a p-point which is almost rapid and not rapid.

Open problems

Question

Is the class of all ultrafilters Tukey above ω^ω Tukey above ω^ω the same as the class of α-almos-rapid ultrafilters?

Question

Is it true that for any two ultrafilters U, V on any cardinals κ, λ, $U \cdot V \equiv_T V \cdot U$?

We can restrict to the case that U, V are on the same cardinal, but the degree of completeness may vary not.

Question

Is it consistent to have two non-Tukey top ultrafilters U, V such that $U \cdot V$ is Tukey top? namely, is the class of non-Tukey top ultrafilters closed under Fubini products?