# APPLICATIONS OF THE MAGIDOR ITERATION TO ULTRAFILTER THEORY

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Abstract. We characterize sums of normal ultrafilters after the Magidor iteration of Prikry forcings over a discrete set of measurable cardinals. We apply this to show that the weak Ultrapower Axiom is not equivalent to the Ultrapower Axiom. We also construct a non-rigid ultrapower and two uniform ultrafilters on different cardinals that have the same ultrapower.

## 1. Introduction

Iterated Prikry forcing was first introduced by Magidor [10] in his seminal *"study on identity crises"* to produce a model of ZFC in which the least measurable cardinal is strongly compact. The rough idea is to iteratively singularize each cardinal $\alpha$ in some set of measurable cardinals $\Delta$ using the Prikry forcing associated with a normal measure on $\alpha$.

Ben-Neria [1] was the first to notice that if Magidor's construction is carried out over the core model $K$, then the normal ultrafilters of the forcing extension can be classified in terms of the normal ultrafilters of $K$. Recently, Ben-Neria's work was substantially extended and generalized by Kaplan [8], who showed, most significantly, that the classification could be carried out even when the ground model is not the core model.

This paper is focused on the special case of Magidor's construction in which the set of measurable cardinals $\Delta$ to be singularized is *discrete* in the sense that it does not contain any of its limit points. In this special case, Kaplan's theorem can be stated as follows:

**Theorem 1.1** (Kaplan). *Suppose $G$ is $V$-generic for a Magidor iteration of Prikry forcings on a discrete set $\Delta \subseteq \kappa$. Then every normal measure $U$ on $\kappa$ in $V$ generates a normal ultrafilter $U^*$ in $V[G]$. Moreover $j_{U^*} \restriction V = i \circ j_U$, where $i$ is an iterated ultrapower of $V_U$.*

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Here and below,

$$j_U : V \to V_U$$

denotes the ultrapower of the universe of sets by $U$.

This paper addresses the question of whether Theorem 1.1 can be generalized to ultrafilters $U$ that are not normal. Although we do prove a slight generalization of Kaplan's result to a collection of measures we call *mild* (see 3.8), the more surprising contribution of this paper is a negative answer to the general question: we show in Section 4 that in the context of Theorem 1.1, a $\kappa$-complete ultrafilter $U$ on $\kappa$ can extend in unexpected ways after a discrete Magidor iteration of Prikry forcings. In particular, $U$ does not generate an ultrafilter in the forcing extension. Moreover, these unexpected extensions answer several natural questions in the theory of ultrafilters.

For example, recall that Kunen's inconsistency theorem states that there is no nontrivial elementary embedding from the universe of sets to itself. Suppose $U$ is a countably complete ultrafilter, and let $V_U$ be the ultrapower of the universe of sets by $U$. Can there be a nontrivial elementary embedding from $V_U$ to itself? By [6, Theorem 3.3, Theorem 4.29], the answer is no if $V = \text{HOD}$ or if $U$ is $\kappa^+$-complete where $\kappa$ is extendible. Nevertheless, we answer the question positively assuming the consistency of a measurable limit of measurable cardinals:

**Theorem 1.2.** *It is consistent with* ZFC *that for some normal ultrafilter $U$, there is an elementary embedding from $V_U$ to itself.*

We then consider the question of ultrafilters with the same ultrapower. Suppose $U$ and $W$ are countably complete ultrafilters such that $V_U = V_W$. Must $U$ and $W$ be isomorphic (that is, Rudin–Keisler equivalent)? Woodin observed that the answer is no [6, Theorem 3.1], though again this is true if $V = \text{HOD}$. We consider the weaker question: must there exist $X \in U$ and $Y \in W$ with $|X| = |Y|$? Again, we show the answer is no after a discrete Magidor iteration of Prikry forcings, assuming the consistency of a limit of measurable cardinals of measurable cofinality:

**Theorem 1.3.** *It is consistent with* ZFC *that there are countably complete uniform ultrafilters $U$ and $W$ on distinct cardinals such that $V_U = V_W$.*

Recall that the Ultrapower Axiom (UA) [5] states that for any two countably complete ultrafilters $U, W$, there are $W' \in V_U$ and $U' \in V_{W'}$ such that $W'$ is a countably complete ultrafilter in $V_U$, $U'$ is a countably complete ultrafilter in $V_W$, $(V_U)_{W'} = (V_W)_{U'}$ and $j_{W'} \circ j_U = j_{U'} \circ j_W$. The Weak Ultrapower Axiom (Weak UA) is the same statement, omitting the requirement that $j_{W'} \circ j_U = j_{U'} \circ j_W$.

The conclusions of the previous theorems are incompatible with UA by [6, Theorem 5.2]. Therefore UA typically becomes false after performing a discrete Magidor iteration of Prikry forcings. Nevertheless, starting from a model of UA containing no measurable cardinals $\delta$ of Mitchell order $2^{2^\delta}$, our main technical result (Theorem 5.7) yields a classification of the lifts of $\kappa$-complete ultrafilters on $\kappa$ to the forcing extension.

This classification suffices to show in certain cases that Weak UA holds in the forcing extension. This answers a question of the second author [5, Question 9.2.4], assuming there is a measurable limit of measurable cardinals:

**Theorem 1.4.** *The Weak Ultrapower Axiom does not imply the Ultrapower Axiom.*

The above results follow from our main technical theorem which is an analysis of all possible extensions of $\kappa$-complete ultrafilters over $\kappa$ whose ultrapower can be factored into a finite iterated ultrapower by normal ultrafilters. The following consequence gives a sense of what this classification entails:

**Theorem 1.5.** *Suppose that $W \in V$ is a $\kappa$-complete ultrafilter over $\kappa$ that can be factored into a finite iteration of normal ultrafilters. Let $G$ be $V$-generic for a discrete Magidor iteration of Prikry forcings. Then in $V[G]$, $W$ extends to at most countably many countably complete ultrafilters.*

*Moreover, if $W^*$ is an extension of $W$, then $j_{W^*} \restriction V = i \circ e \circ j_W$, where $e : V_W \to N$ is a finite external iteration of $V_W$ by normal measures and $i$ is the complete iteration of $N$ by $e(j_W(\vec{U}))$.*

The previous theorem also gives a sense in which our classification generalizes Kaplan's result.

This paper is organized as follows:

- In Section §2 we prove some preliminary results regarding ultrapowers and the discrete Magidor iteration.
- In Section §3 we study the complete iteration by a sequence of normal measures and how it relates to the restriction of an ultrapower of the generic extension to the ground model.
- In Section §4 we describe an ultrafilter which has many extensions to the generic extension of the discrete Magidor iteration and characterize its extensions. Also, we provide the example of a non-rigid ultrapower.
- In Section §5 we prove our characterization of lifts of ultrafilters to the generic extension by the discrete Magidor iteration.
- In Section §6 we prove some applications of our characterization: a model of weak UA which fails to satisfy UA, and an example of the same ultrapower by ultrafilters on different cardinals.
- In Section §7 we present some related open problems.

## 2. Preliminaries

2.1. **Derived ultrafilters and commuting squares.** Let $P$ be a transitive model of set theory and let $U$ be a (possibly external) $P$-ultrafilter over a set $X \in P$.[1] We will always assume that $U$ is countably complete; namely, for any countable $\mathcal{A} \subseteq U$, $\bigcap \mathcal{A} \neq \emptyset$. Denote by $j_U^P : P \to P_U$ the ultrapower of $P$ by $U$ using function $f : X \to P$ in $P$. We will suppress the superscript $P$ from $j_U^P$ whenever there is no ambiguity. Since $U$ is countably complete, $P_U$ is well-founded and we identify $P_U$ with its transitive collapse.

Let $j : P \to Q$ be an elementary embedding of transitive models of set theory, $X \in P$, and $a \in j(X)$. The *$P$-ultrafilter on $X$ derived from $j$ and $a$* is the set

$$U = \{A \in P(X) \cap P \mid a \in j(A)\}$$

(The underlying set $X$ is typically suppressed.) It is well known that $U$ is a $P$-ultrafilter and that the map $k : P_U \to Q$ defined by $k([f]_U) = j(f)(a)$ is the unique elementary embedding mapping $[id]_U$ to $a$ such that $k \circ j_U = j$. The following generalization of this fact will be used implicitly in many calculations below:

**Lemma 2.1** (Shift lemma)**.** *Suppose $i : P \to Q$ is an elementary embedding, $W$ is a $Q$-ultrafilter, and $U = i^{-1}[W]$. Then the embedding $k : P_U \to Q_W$ defined by $k([f]_U) = [i(f)]_W$ is well-defined and elementary. Moreover, it is the unique embedding mapping $[id]_U$ to $[id]_W$ such that the following diagram commutes:*

$$\begin{array}{ccc} Q & \xrightarrow{j_W} & Q_W \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle k} \\ P & \xrightarrow[j_U]{} & P_U \end{array}$$

*Proof.* Note that $U$ is the $P$-ultrafilter derived from $j_W \circ i$ and $[id]_W$. Moreover, $k$ is the associated factor map, and the lemma follows. □

**Lemma 2.2.** *Suppose that $D$ is a $P$-ultrafilter $j : P \to M$, $j' : P_D \to M'$, and $k : M \to M'$ is such that $M' = \mathrm{Hull}^{M'}(\mathrm{rng}(k) \cup \mathrm{rng}(j'))$ and the following diagram commutes:*

$$\begin{array}{ccc} P_D & \xrightarrow{j'} & M' \\ {\scriptstyle j_D}\uparrow & & \uparrow{\scriptstyle k} \\ P & \xrightarrow[j]{} & M \end{array}$$

*Then $M_{D'} = M'$ and $k = j_{D'}$, where $D'$ is the $M$-ultrafilter derived from $k$ and $j'([id]_D)$.*

[1] That is, $U$ is an ultrafilter on the Boolean algebra $P(X) \cap P$.

*Proof.* Note that there is a factor map $k' : M_{D'} \to M'$ defined by $k'([f]_{D'}) = k(f)(j'([id]_D))$ such that $k' \circ j_{D'} = k$. To see that $k'$ is onto, we have

$$\begin{aligned} M' &= \mathrm{Hull}^{M'}(\mathrm{rng}(k) \cup \mathrm{rng}(j')) \\ &= \mathrm{Hull}^{M'}(\mathrm{rng}(k) \cup \mathrm{rng}(j' \circ j_D) \cup \{j'([id]_D)\}) \\ &= \mathrm{Hull}^{M'}(\mathrm{rng}(k) \cup \{j'([id]_D)\}) \subseteq \mathrm{rng}(k') \end{aligned}$$

□

**Corollary 2.3.** *If $k : M \to M'$, $M' = \mathrm{Hull}^{M'}(\mathrm{rng}(k) \cup \{a\})$, and $a \in k(X)$ for some $X \in M$, then $M_D = M'$ and $k = j_D$ where $D$ is the $M$-ultrafilter over $X$ derived from $a$ and $k$.*

*Proof.* This is a special case of the previous lemma where $j = id$ and $j'$ is the factor embedding. □

2.2. **Discrete iteration of Prikry-type forcings.** A set of ordinals $\Delta$ is called *discrete* if it contains none of its accumulation points; that is, for every $\alpha \in \Delta$, $\sup(\Delta \cap \alpha) < \alpha$. In particular, $\Delta$ is an extremely thin non-stationary set, and so are all its restrictions to $\alpha < \sup(\Delta)$; similarly, if $\Delta \subseteq \kappa$ is discrete, then for every $\kappa$-complete ultrafilter $U$ over $\kappa$, $\kappa \notin j_U(\Delta)$. Note that if $\kappa$ is the least measurable limit of measurable cardinals, then the set of measurables below $\kappa$ is discrete.

For the rest of this subsection we fix a discrete set $\Delta \subseteq \kappa$ of measurable cardinals and a sequence $\vec{U} = \langle U_\alpha \mid \alpha \in \Delta \rangle$ such that $U_\alpha$ is a normal measure of Mitchell order 0 over $\alpha$. We define the *Magidor support iteration* $\langle \mathbb{P}_\alpha(\vec{U}), \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ associated with $\vec{U}$ as follows.

**Definition 2.4.** Define inductively for each $\alpha \leq \kappa$, a condition $p$ in $\mathbb{P}_\alpha$ is a function $p$ with $\mathrm{dom}(p) = \alpha$ such that:

1. for every $\beta < \alpha$, $p \restriction \beta \in \mathbb{P}_\beta$.
2. for every $\beta < \alpha$, $p \restriction \beta \Vdash_{\mathbb{P}_\beta} p(\beta) \in \underset{\sim}{Q}_\beta$, where $\underset{\sim}{Q}_\beta$ is trivial, unless $\beta \in \Delta$, in which case $\underset{\sim}{Q}_\beta$ is a $\mathbb{P}_\beta$-name for the usual Prikry forcing $Pr(\underset{\sim}{\bar{U}}_\beta)$ from [12], where $\bar{U}_\beta$ is the filter generated by $U_\beta$ in $V^{\mathbb{P}_\beta}$(see Proposition 2.6(2)).
3. there is a finite set $b_p$ such that for every $\beta \in \alpha \setminus b_p$, $p \restriction \beta \Vdash_{\mathbb{P}_\beta} p(\beta) \leq^* \emptyset$.

We define $p \leq q$ iff for every $\beta < \alpha$, $p \restriction \beta \Vdash_{\mathbb{P}_\beta} p(\beta) \leq q(\beta)$. Also define $p \leq^* q$ iff for every $\beta < \alpha$, $p \restriction \beta \Vdash_{\mathbb{P}_\beta} p(\beta) \leq^* q(\beta)$.

*Remark* 2.5. Note that since the set $\Delta$ is discrete, a Magidor support here is always going to be non-stationary. Hence the forcing above inherits the properties of both the non-stationary and Magidor support iterations of Prikry-type forcings on the set $\Delta$.

**Proposition 2.6.** *For each $\alpha \leq \kappa$,*

1. *$\mathbb{P}_\alpha$ is $\alpha^+$-cc.*

(2) *If $\alpha \in \Delta$, $U_\alpha$ generates a normal ultrafilter in $V^{\mathbb{P}_\alpha}$.*
(3) *The set of all conditions $p$ such that for every $\beta \in \Delta \cap \alpha$, $\underset{\sim}{A}^p_\beta$ is a canonical name for a set in $U_\beta$ is $\leq^*$-dense in $\mathbb{P}_\alpha$.*
(4) *The set of all conditions $p$ such that for every $\beta \in \Delta \cap \alpha$, $p(\beta) = \check{q}$ for some $q \in Pr(U_\beta)$ is $\leq$-dense in $\mathbb{P}_\alpha$.*

By the previous proposition, we obtain the following corollary:

**Corollary 2.7.** *For every $\alpha \leq \kappa$, $\mathbb{P}_\alpha$ is forcing equivalent to the Magidor support product*[2] *$\prod_{\beta \in \Delta \cap \alpha} Pr(U_\beta)$.*

We shall further consider $\mathbb{P}_\kappa$ as the product $\prod_{\beta \in \Delta \cap \alpha} Pr(U_\beta)$. The Prikry Lemma can be derived more generally for iterations of Prikry-type forcings (see [3]).

**Lemma 2.8** (Prikry Lemma)**.** *For every $p \in \mathbb{P}_\alpha$, and every sentence $\sigma$ in the forcing language for $\mathbb{P}_\kappa$, there is $p^* \leq^* p$ such that $p^* \Vdash \sigma$.*

2.3. **On ground model ultrafilters that generate ultrafilters in the extension.** In this subsection, we provide a version of Theorem 1.1 which provides slightly more information in the context of discrete sequences of measures.

**Definition 2.9.** Let $\Sigma \subseteq \lambda$ be unbounded. A $\lambda$-complete ultrafilter $U$ over $\lambda$ is called $\Sigma$-*mild* if there is a function $f : \lambda \to \lambda$ such that

$$[id]_U \leq j_U(f)(\lambda) < \min(j_U(\Sigma) \setminus \lambda)$$

Note that if $U$ is a normal measure with $\Sigma \notin U$, then $U$ is $\Sigma$-mild. Also, every $\Sigma$-mild ultrafilter is a $p$-point.

**Theorem 2.10.** *If $U$ is a $(\Delta \cap \lambda)$-mild ultrafilter over a cardinal $\lambda \notin \Delta$, then $U$ generates a $\lambda$-complete ultrafilter in $V[G]$.*

*Proof.* By Gitik and Kaplan [4, Prop. 2.6& Prop. 2.7], since $\lambda$ and $\lambda^+$ remain regular in $V[G]$, it suffices to prove that $j_U[G]$ decides all statements of the form $[id]_U \in j_U(\dot{A})$, where $\dot{A}$ is a $\mathbb{P}_\kappa$-name for a subset of $\lambda$. Given such a name $\dot{A}$, we may assume that it is a $\mathbb{P}_\kappa \restriction \lambda$-name. Working in $M_U[G]$, let $\dot{A}_0$ be the $\mathbb{P}_{(\lambda, j_U(\lambda))}$-name obtained from $\dot{A}$. By the Prikry property, there is a condition $q \leq 1_{\mathbb{P}_{(\lambda, j_U(\lambda))}}$ such that $q || [id]_U \in \dot{A}_0$ and therefore there is $p \in G$ such that $p \Vdash q || [id]_U \in j_U(\dot{A})$, namely $(p, q) || j_U(\dot{A})$. Let us show that there is $p' \in G$ such that $j_U(p') \leq (p, q)$. Since the iteration is the same as the product, we may assume that $q = \langle A_\gamma \mid \gamma \in j_U(\Delta) \cap [\lambda, j_U(\lambda)) \rangle$. Let $\alpha \mapsto \langle A^{(\alpha)}_\gamma \mid \gamma \in \Delta \cap [\pi(\alpha), \lambda) \rangle$ represent $q$ in $j_U$ (here $\pi$ is the function representing $\lambda$ is $M_U$).

Let us proceed with a density argument. Let $p \in \mathbb{P}_\kappa$. We shrink $p$ in the interval $[\gamma_0, \lambda)$, where $\gamma_0 = \max(b_p \cap \lambda) + 1$. For $\gamma \geq \gamma_0$, $\gamma \in \Delta \cap \lambda$, define

[2] That is, a condition is a function $p$ with $\mathrm{dom}(p) = \Delta$, for every $\alpha \in \Delta$, $p(\alpha) \in Pr(U_\alpha)$, and $\{\alpha \mid p(\alpha) \text{ is not pure}\}$ is finite.

$A^0_\gamma = \Delta_{\delta<\gamma} A^{(\delta)}_\gamma$. Moreover, if $f(\sup(\Delta\cap\gamma)) \geq \gamma$ we let $A^*_\gamma = A^0_\gamma \setminus \sup(\Delta\cap\gamma)$ and if $f(\sup(\Delta\cap\gamma)) < \gamma$ we let

$$A^*_\gamma = A^0_\gamma \setminus \max\{f(\sup(\Delta\cap\gamma)), \sup(\Delta\cap\gamma)\}.$$

We shrink $p$ so that $A^p_\gamma \subseteq A^*_\gamma$ for $\gamma \in \Delta\setminus\gamma_0$, and by genericity we can find such a $p^* \in G$ such that $p^* \leq p$. Then $j_U(p^*) \restriction \lambda = p^* \leq p$ and $j_U(p^* \restriction [\lambda,\kappa)) \leq j_U(p \restriction [\lambda,\kappa))$. For $\gamma \in j_U(\Delta)\cap[\lambda, j_U(\kappa))$, we note that $\lambda \notin j_U(\Delta)$ by our assumption that $\lambda \notin \Delta$ and $\Delta$ is discrete. We need to argue that $A^{j_U(p^*)}_\gamma \subseteq A_\gamma$. Denote by

$$j_U(\alpha \mapsto \langle A^{(\alpha)}_\gamma \mid \gamma \in \Delta\cap[\pi(\alpha),\lambda)\rangle)(\beta) = \langle A_{\beta,\gamma} \mid \gamma \in j_U(\Delta)\cap[j_U(\pi)(\beta), j_U(\lambda))\rangle.$$

Then by definition, $A_{[id]_U,\gamma} = A_\gamma$ for every $\gamma \in j_U(\Delta)\cap[\lambda, j_U(\lambda))$. If $\gamma = \min(j_U(\Delta)\setminus\lambda)$, then $j_U(f)(\sup(j_U(\Delta)\cap\lambda)) = j_U(f)(\lambda) \geq [id]_U$ and we have

$$A^{j_U(p^*)}_\gamma \subseteq \Delta_{\beta<\gamma} A_{\beta,\gamma} \setminus j_U(f)(\lambda)+1 \subseteq A_{[id]_U,\gamma} = A_\gamma.$$

If $\gamma > \min(j_U(\Delta)\setminus\lambda)$, then $\sup(j_U(\Delta)\cap\gamma) \geq \min(j_U(\Delta)\cap\gamma) \geq [id]_U$, hence we still have $A^{j_U(p^*)}_\gamma \subseteq A_\gamma$. It follows that $j_U(p^*) \leq p^\frown q$ as wanted. □

**Example 2.11.** The discreteness of the set $\Delta$ is essential here; without it, even 0-order normal measures might not generate ultrafilters. A counterexample was pointed out to us by Kaplan. Consider the Magidor iteration from [10]. Take any normal measure $U_1$ over $\kappa$ of Mitchell order 1 in the ground model. Using Kaplan's result [8], $U^\times_1$ is a normal measure in $V[G]$ (note that $U^\times_1$ does not extend $U_1$, as $U_1$ does not extend to a normal ultrafilter in $V[G]$ by [8, Lemma 2.3]). Let $U_0 = U_1\cap V$, then $U_0$ is a normal ultrafilter in $V$ (again, this follows from Kaplan's result). Hence $U_0$ must be of order 0. Consider the ultrafilter $U^*_0$ from [8, Lemma 2.3], then $U^*_0$ is a normal ultrafilter in $V[G]$ and we claim that $U^*_0 \neq U^\times_1$ which yield two distinct extensions of $U_0$ to normal ultrafilters in $V[G]$. Indeed, since $U_1$ is of order 1, the set $\mathcal{M}$ of all measurable below $\kappa$ is in $U_1$, and therefore, by definition $d[\mathcal{M}] \in d_*(U_1) = U^\times_1$ where $d(\delta)$ is the first element of the Prikry sequence associated with $\delta$. Also, as $U$ is of order 0, it is possible to show that $d[\mathcal{M}] \notin U^*_0$ hence $U^\times_1 \neq U^*_0$.

**Example 2.12.** In sub-section 4.1, we will give an example where unique lifting fails but still $[id]_U < \min(j_U(\Delta)\setminus\kappa)$. Hence the assumption regarding the existence of the function $f$ is indeed necessary. Note that if $[id]_U$ is below the first $M_U$-Ramsey cardinal (for example) above $\lambda$, then $U$ is $\Delta$-mild. By the counterexample of Section 4.1, $[id]_U$ being below the first $M_U$-measurable cardinal does not suffice for $U$ to be $\Delta$-mild (or even for unique lifting).

## 3. The complete iteration

One aspect of understanding the lifts $\overline{W}$ of an ultrafilter $W$ to a generic extension $V[G]$ is to analyze the factor map $k : V_W \to j^{V[G]}_{\overline{W}}(V)$ defined by $k([f]_W) = [f]^{V[G]}_{W^*}$. In the case of a generic extension by an iteration of Prikry forcings, the factor map can often be described in terms of the complete iteration:

**Definition 3.1.** Let $\vec{U} = \langle U_\delta \mid \delta \in \Delta\rangle$ be a sequence such that $U_\delta$ is a normal measure on $\delta$ and $\Delta$ is any set of measurables. Let us define, by transfinite recursion, the *complete iteration* associated with the sequence $\vec{U}$. (We usually omit the superscript $\vec{U}$ from $i^{\vec{U}}$, and we denote $\theta$ by $\infty$ when the length of the iteration does not play a significant role.) This will be an iterated ultrapower denoted by $\langle i^{(\vec{U})}_{\alpha,\beta} \mid \alpha \leq \beta \leq \theta\rangle$. Simultaneously, we will define sets of ordinals $\langle s^\alpha_\delta \mid \delta \in i_{0,\alpha}(\Delta)\rangle$.

For $\alpha < \theta$, let $W_\alpha = i_\alpha(\vec{U})_{\delta_\alpha}$ where $\delta_\alpha \in i_\alpha(\Delta)$ is the minimal $\delta$ such that the set $s^\alpha_\delta$ is finite. Let $N_{\alpha+1} = (N_\alpha)_{W_\alpha}$, and let $i_{\alpha,\alpha+1} : N_\alpha \to N_{\alpha+1}$ be the ultrapower embedding associated with $W_\alpha$. Define $s^{\alpha+1}_\beta = s^\alpha_\beta$ unless $\beta = i_{\alpha,\alpha+1}(\delta_\alpha)$ in which case $s^{\alpha+1}_{i_{\alpha,\alpha+1}(\delta)} = s^\alpha_{\delta_\alpha} \cup \{\delta_\alpha\}$. At limit steps, we take direct limits (of both embeddings and the sequences of $s^\alpha_\delta$'s). The sequence $\langle s^\theta_\alpha \mid \alpha \in i_{0,\theta}(\Delta)\rangle$ is called the *sequence of sets of indiscernibles* associated with the complete iteration of $\vec{U}$.

For example, if $\Delta$ consists of a single measurable cardinal, then the complete iteration is simply the $\omega^{\text{th}}$ iterated ultrapower of the only normal measure in $\vec{U}$, and the sequence of sets of indiscernibles consists of the corresponding sequence of iteration points.

**Proposition 3.2.** *Let $i : V \to N$ be the complete iteration of $\vec{U}$, where* $\text{dom}(\vec{U}) = \Delta$ *is discrete. Then:*

1. *For every $\alpha$, $\delta_{\alpha+1} = i_{\alpha,\alpha+1}(\delta_\alpha)$.*
2. *For each $\gamma \in \Delta$, there is a unique $\alpha = \alpha(\gamma)$ such that $\delta_\alpha = \gamma$.*
3. *For each $\gamma \in \Delta$, $i_{0,\alpha}(\gamma) = \gamma$, and $i(\gamma) = i_{\alpha,\alpha+\omega}(\gamma)$, where $\alpha = \alpha(\gamma)$.*

*Proof.* For (1), we have that $i_\alpha(\Delta) \cap \delta_\alpha$ is bounded in $\delta_\alpha$, and therefore

$$i_{\alpha+1}(\Delta) \cap i_{\alpha,\alpha+1}(\delta_\alpha) = i_\alpha(\Delta) \cap \delta_\alpha.$$

Hence every $\delta \in i_{\alpha+1}(\Delta) \cap i_{\alpha,\alpha+1}(\delta_\alpha)$ has been used infinitely many times and $\delta_{\alpha+1} \geq i_{\alpha,\alpha+1}(\delta_\alpha)$. By definition of $\delta_\alpha$, equality must hold.

For (2) and (3), fix any $\gamma \in \Delta$, then $\gamma$ is measurable, it is routine to show inductively on $\beta$, if $\delta_\beta < \gamma$, then $i_{0,\beta}(\gamma) = \gamma$. Since the sequence $\delta_\alpha$ is strictly increasing, and since $\Delta$ is discrete, after less than $\gamma$-many steps of the iteration we reach a stage $\alpha$, such that $\gamma = \delta_\alpha$. This means that the measure $i_{0,\alpha}(\vec{U})_\gamma$ was not used before stage $\alpha$, and every previous measure was used infinitely many times already. By (1), $\delta_{\alpha+n} = i_{\alpha,\alpha+n}(\gamma)$, for every $n < \omega$. At stage $\alpha + \omega$ we get:

$$i_{\alpha,\alpha+\omega}(i_{0,\alpha}(\Delta)) = [i_{0,\alpha}(\Delta) \cap \gamma] \cup i_{\alpha,\alpha+\omega}(i_{0,\alpha}(\Delta) \setminus \gamma)$$

So the minimal element which can potentially be applied now is $i_{\alpha,\alpha+\omega}(\gamma)$, however, we have already used this measurable $\omega$-many times, so by the definition of the complete iteration we must go to the next element which makes the critical point of $i_{\alpha+\omega,\theta}$ greater than $i_{\alpha,\alpha+\omega}(\gamma)$. We conclude that $i_{\alpha+\omega,\theta}(i_{\alpha,\alpha+\omega}(\gamma)) = i_{\alpha,\alpha+\omega}(\gamma)$ and therefore $i(\gamma) = i_{\alpha,\alpha+\omega}(\gamma)$ as wanted. Also we see that for every $\alpha$,

$$i(\Delta) = i_{0,\alpha}(\Delta) \cap \delta_\alpha \cup \{i_{\alpha,\alpha+\omega}(\delta_\alpha)\} \cup i(\Delta) \setminus i_{\alpha,\alpha+\omega}(\delta_\alpha)$$

□

**Definition 3.3.** Given a sequence of sets $\vec{s} = \langle s_\alpha \mid \alpha \in \Delta\rangle$ such that for each $\alpha \in \Delta$, $s_\alpha$ has order type $\omega$ we associate a filter $G_{\vec{s}}$ on $\mathbb{P}_\kappa$ consisting of all conditions $p$ such that for each $\alpha$, $t^p_\alpha \sqsubseteq s_\alpha$ and $s_\alpha \setminus t^p_\alpha \subseteq A^p_\alpha$.

**Lemma 3.4.** *Let $\vec{U}$ be a sequence of normal measures on a discrete set $\Delta$ with $\sup(\Delta) = \kappa$. Let $i = i_{0,\theta} : V \to N$ be the complete iteration of $V$ by $\vec{U}$, and $\vec{s} = \langle s^\theta_\alpha \mid \alpha \in i(\Delta)\rangle$ be the sequence of sets of indiscernibles, and let $G = G_{\vec{s}}$. Then the following hold:*

1. *$G$ is $N$-generic for $i(\mathbb{P}_\kappa)$.*
2. *If $p$ is a pure condition then $i(p) \in G$.*
3. *Every element of $N$ is $\Sigma_2$-definable in $N[G]$ from parameters in $i[V] \cup \{G\}$.*
4. *$N[G]$ is closed under $\delta$-sequences where $\delta = \min(\Delta)$.*

*Proof.* The genericity of $G$ is due to Fuchs [2]. For the second item, if $p$ is pure, then by induction we can prove that for every $\xi \leq \theta$, $s^\xi_\alpha \subseteq A^{i_{0,\alpha}(p)}_\alpha$. This implies in particular that $s^\theta_\alpha \subseteq A^{i(p)}_\alpha$, as wanted. Suppose this was true for $\xi$, since $i_{\xi,\xi+1}$ is the ultrapower by the normal $W_\xi$ on $\delta_\xi$, and $A^{i_{0,\xi}(p)} \in W_\xi$, we conclude that $\delta_\xi \in A^{i_{\xi,\xi+1}(p)}_{\delta_\xi}$. Since below $\delta_\xi$, things do not change, we get that $s^{\xi+1}_{i_{\xi,\xi+1}(\delta_\xi)} \subseteq A^{i_{\xi,\xi+1}(p)}_{\delta_\xi}$. Note that above $\xi$, all the sequences are empty at this stage of the iteration.

For the third item, let $H$ be the class of all elements of $N$ that are $\Sigma_2$-definable in $N[G]$ from parameters in $i[V] \cup \{G\}$. Note that $H \prec N$, and let $\pi : M \to N$ be the inverse of the transitive collapse of $H$. To establish that $N = H$, we will show that $\pi$ is surjective. Since every element of $N$ is of the form $i(f)(\kappa_1, ..., \kappa_m)$ for some $f : [\kappa]^m \to V$, where $\kappa_1, ..., \kappa_m \in \bigcup \vec{s}$, it suffices to prove that $\bigcup \vec{s} \subseteq \text{rng}(\pi)$. In turn, it suffices to prove that $i(\Delta) \subseteq \text{rng}(k)$, since any element in $\bigcup \vec{s}$ is a Prikry point associated with one of the elements of $i(\Delta)$ and therefore is definable from $G$ and $i(\Delta)$.

Suppose towards contradiction that $\zeta$ is the minimal element of $i(\Delta)$ not in $\text{rng}(\pi)$. Since $i(\Delta)$ is discrete, then $\gamma = \sup(i(\Delta) \cap \zeta) < \zeta$. By the minimality of $\zeta$, all the generators of $i$ below $\zeta$ are in $\text{rng}(\pi)$ and therefore $\text{crit}(\pi) \geq \gamma$. Since $\gamma \notin \bigcup \vec{s}$ (as $i(\Delta)$ is discrete), $\gamma = i(f)(\kappa_1, ..., \kappa_m)$ for some $\kappa_1, ..., \kappa_m < \gamma$ and therefore $\gamma \in \text{rng}(\pi)$. Hence $\text{crit}(\pi) > \gamma$. Let $\bar{\zeta} = \min(\pi^{-1}(\Delta) \setminus \gamma)$. By elementarity of $\pi$, $\pi(\bar{\zeta}) = \zeta$, contradiction.

The final item follows from the previous one using the standard proof that the ultrapower of $V$ by a $\delta$-complete ultrafilter is closed under $\delta$-sequences.

(In fact, one can view $N[G]$ as a Boolean ultrapower of $V$ and apply [7, Corollary 26].) Fix a $\delta$-sequence of ordinals $\langle\beta_\alpha\rangle_{\alpha<\delta}$. By the previous item, each $\beta_\alpha$ is definable in $N[G]$ from the parameters $i(a_\alpha)$ and $G$ via some $\Sigma_2$ formula $\phi_\alpha$. Note that

$$\langle i(a_\alpha), \phi_\alpha \mid \alpha < \delta\rangle = i(\langle a_\alpha, \phi_\alpha \mid \alpha < \delta\rangle) \restriction \delta \in N$$

Therefore $\langle\beta_\alpha \mid \alpha < \delta\rangle \in N[G]$. □

The following proposition shows that any ultrafilter used in the complete iteration gives rise to an elementary embedding of the target model of the complete iteration.

**Proposition 3.5.** *If $i = i_{0,\theta} : V \to N$ is the complete iteration $V$ by $\vec{U}$, $\vec{s}$ is the associated sequence of sets of indiscernibles, and $U$ is an ultrafilter on $\delta$ applied at some stage of the complete iteration, then $j_U^{N[G_{\vec{s}}]} \circ i = i$ and $j_U^{N[G_{\vec{s}}]}(\vec{s}) = \vec{s} \setminus \{\delta\}$.*

*Proof.* We first consider the special case that $\delta$ is the least element of the domain of $\vec{U}$. Note that $j_U^{N[G_{\vec{s}}]} = j_U^V \restriction N[G_{\vec{s}}]$ because $N[G_{\vec{s}}]$ is closed under $\delta$-sequences. Therefore

$$j_U^{N[G_{\vec{s}}]} \circ i = j_U \circ i = j_U(i) \circ j_U = i^U \circ j_U$$

where $i^U : V_U \to N$ denotes the complete iteration of $V_U$ by $j_U(\vec{U})$. By the definition of the complete iteration, $i^U \circ j_U = i$, which proves that $j_U^{N[G_{\vec{s}}]} \circ i = i$. Similarly, $j_U^{N[G_{\vec{s}}]}(\vec{s})$ is the sequence of sets of indiscernibles associated with the complete iteration of $V_U$ by $i_U(\vec{U})$, which is $\vec{s} \setminus \{\delta\}$.

To prove the claim in general, suppose $U$ is the ultrafilter used at stage $\alpha$ of the complete iteration. Then $i = i_{\alpha\infty} \circ i_{0\alpha}$ where $i_{0\alpha} : V \to N_\alpha$ is an initial segment of the complete iteration $V$ by $\vec{U}$ and $i_{\alpha\infty} : N_\alpha \to N$ is the complete iteration of $N_\alpha$ by $\vec{W} = i_{0\alpha}(\vec{U}) \restriction [\delta, \infty)$. Let $\vec{s}^{\,0\alpha}$ denote the sequence of sets of indiscernibles associated with $i_{0\alpha}$ and let $\vec{s}^{\,\alpha\infty}$ denote the sequence of sets of indiscernibles associated with $i_{\alpha\infty}$. Then $G_{\vec{s}^{0\alpha}} \times G_{\vec{s}^{\alpha\infty}} = G_{\vec{s}}$.

The special case of the claim proved above, applied in $N_\alpha$ to $U$ and $\vec{W}$, implies that $j_U^{N[G_{\vec{s}^{\alpha\infty}}]} \circ i_{\alpha\infty} = i_{\alpha\infty}$ and $j_U^{N[G_{\vec{s}^{\alpha\infty}}]}(\vec{s}^{\,\alpha\infty}) = \vec{s}^{\,\alpha\infty} \setminus \{\delta\}$. This yields the full claim almost immediately, except that we have to verify that $j_U^{N[G_{\vec{s}^{\alpha\infty}}]}$ is the restriction of $j_U^{N[G_{\vec{s}}]}$ to $N[G_{\vec{s}^{\alpha\infty}}]$. But $j_U^{N[G_{\vec{s}}]} = j_U^{N[G_{\vec{s}^{0\alpha}} \times G_{\vec{s}^{\alpha\infty}}]}$, which extends $j_U^{N[G_{\vec{s}^{\alpha\infty}}]}$ by the proof of the classical Lévy-Solovay theorem [9]. □

3.1. **The canonical extension of an ultrafilter.** Suppose that in $V$, $W$ is a $\kappa$-complete ultrafilter on a set $X$. Using a construction due to Mitchell [11], we will define a $\kappa$-complete ultrafilter $W^*$ of $V[G]$ extending $W$.

Let $j : V \to M$ be the ultrapower embedding associated with $W$. Let $\mathbb{Q} = \mathbb{P}_{>\kappa}$, so that $j(\mathbb{P}_\kappa)$ can be naturally identified with $\mathbb{P}_\kappa \times \mathbb{Q}$. To lift $j$ to $V[G]$, one might try to produce an $M$-generic $H$ on $\mathbb{Q}$ such that $j[G] \subseteq$

$G \times H$. If $\Delta$ is unbounded in $\kappa$, however, no such $M$-generic $H$ can exist in $V[G]$: the ordinal $\delta = \min(j(\Delta) \cap (\kappa, j(\kappa)))$ has uncountable cofinality in $V[G]$, so in $V[G]$, there is no cofinal $\omega$-sequence in $\delta$.

Instead, we will produce an inner model $N$ and an elementary embedding $i : M \to N$ such that there is an $N$-generic filter $H$ on $i(\mathbb{Q})$ with $i \circ j[G] \subseteq G \times H$. Then by the Silver lifting criterion, $i \circ j$ lifts to an elementary embedding from $V[G]$ to $N[G \times H]$.

Let $i : M \to N$ be the complete iteration of $M$ via the sequence $j(\vec{U}) \restriction (\kappa, j(\kappa))$. Let $\vec{s}$ be the sequence of sets of indiscernibles associated with the complete iteration, and let $G_{\vec{s}}$ be the associated $N$-generic filter on $i(\mathbb{Q})$ (Definition 3.3).

Note that $G$ is an $N[G_{\vec{s}}]$-generic filter on $\mathbb{P}_\kappa$, and so by the Product Lemma, $G \times G_{\vec{s}}$ is an $N$-generic filter on $\mathbb{P}_\kappa \times \mathbb{Q}$. We must verify that $i \circ j[G] \subseteq G \times G_{\vec{s}}$. For each $p \in \mathbb{P}_\kappa$, $j(p)$ has the form $(p, q)$ where $q \in \mathbb{Q}$ is a *pure condition*, or in other words, a direct extension of $1_{\mathbb{Q}}$, and therefore $i \circ j(p) = (p, i(q))$. For every pure condition $q \in \mathbb{Q}$, $i(q)$ belongs to $H$, by Lemma 3.4. Hence $i \circ j(p) = (p, i(q)) \in G \times G_{\vec{s}}$.

By the Silver lifting criterion, let $j^* : V[G] \to N[G \times G_{\vec{s}}]$ be the unique lift of $i \circ j$ such that $j^*(G) = G \times G_{\vec{s}}$.

**Definition 3.6.** The *canonical extension* of $W$ to $V[G]$ is the $V[G]$-ultrafilter $W^*$ on $X$ derived from $j^*$ using $i([\mathrm{id}]_W)$.

The next proposition shows $N[G \times G_{\vec{s}}] = V[G]_{W^*}$ and $j^* = j_{W^*}$.

**Proposition 3.7.** *Suppose $W$ is a $\kappa$-complete ultrafilter and $W^*$ is the canonical extension of $W$ to $V[G]$. Then $j_{W^*} \restriction V = i \circ j_W$ where $i : V_W \to N$ is the complete iteration of $V_W$ associated with $j_W(\vec{U}) \restriction (\kappa, j_W(\kappa))$. Moreover, $j_{W^*}(G) = G \times G_{\vec{s}}$ where $\vec{s}$ is the sequence of sets of indiscernibles associated with the complete iteration.*

*Proof.* The proposition follows from Corollary 2.3 once we show that

$$N[G \times G_{\vec{s}}] = \mathrm{Hull}^{N[G \times G_{\vec{s}}]}(i \circ j_W[V] \cup \{G, G_{\vec{s}}, i([id]_W)\}).$$

This in fact implies directly that $j_{W^*} = j^*$. To see the above, apply the third item of Lemma 3.4 inside of $M$ we see that

$$N[G_{\vec{s}}] = \mathrm{Hull}^{N[G_{\vec{s}}]}(i \circ j_W[V] \cup \{G_{\vec{s}}, i([id]_W)\}).$$

Then we finish by noting that being a forcing extension of $N[G_{\vec{s}}]$, $N[G \times G_{\vec{s}}]$ is the hull (in itself) of $N[G_{\vec{s}}] \cup \{G\}$. □

In the case that $W$ is $\Delta$-mild, Theorem 2.10 shows that $W^*$ is the unique extension of $W$ in $V[G]$ and therefore Proposition 3.7 can be used to analyze the ultrapower of the ultrafilter generated by $W$ in the generic extension. This is a slight generalization of Kaplan's theorem stated in the introduction. Although this is just the combination of Theorem 2.10 and Proposition 3.7, we record it for future use.

**Corollary 3.8.** *Let $W$ be a $\Delta$-mild ultrafilter in $V$ and let $G \subseteq \mathbb{P}_\kappa$ be $V$-generic. In $V[G]$, $W$ generates its own canonical extension $W^*$ and $j_{W^*} \restriction V = i \circ j_W$, where $i : V_W \to N$ is the complete iteration of $M$ associated with $j(\vec{U}) \restriction (\kappa, j(\kappa))$. Moreover, $j_{W^*}(G) = G \times G_{\vec{s}}$ where $\vec{s}$ is the sequence of sets of indiscernibles associated with the complete iteration.*

3.2. **Mitchell's lemma.** Next, we prove a variation of Mitchell's lemma from [11]:

**Lemma 3.9.** *Let $\vec{U}$ be a sequence of normal measures on a discrete set $\Delta$ with $\sup(\Delta) = \kappa$. Let $i = i_{0,\theta} : V \to N$ be the complete iteration of $V$ by $\vec{U}$, and $\vec{s} = \langle s^\theta_\alpha \mid \alpha \in i(\Delta)\rangle$ be the sequence of sets of indiscernibles, and let $G = G_{\vec{s}}$.*

*Suppose that $W$ is a normal ultrafilter on $\kappa$. Let $W^*$ be the $N[G_{\vec{s}}]$-ultrafilter generated by $i(W)$. Then $j_{W^*} = j_W \restriction N[G_{\vec{s}}]$. Moreover, the canonical factor map $k : N_{i(W)} \to j_{W^*}(N)$ is given by the complete iteration of $N_{i(W)}$ by $i(j_W(\vec{U})) \restriction (\kappa, j_W(\kappa))$.*

*Proof.* We first show that $N[G_{\vec{s}}]_{W^*} = j_W(N[G_{\vec{s}}])$. Let $i' : V_W \to N'$ be the complete iteration of $V_W$ by $j_W(\vec{U})$ and let $\vec{t}$ be the sequence of sets of indiscernibles associated with $i'$. By elementarity, $j_W(N[G_{\vec{s}}]) = N'[G_{\vec{t}}]$. Also, since $\text{crit}(j_W) = \kappa$, $j_W(\vec{U}) \restriction \kappa = \vec{U}$ and thus we can decompose

$$i' = i'_{\kappa, j_W(\kappa)} \circ i \restriction V_W.$$

Here $i'_{\kappa,j_W(\kappa)}$ is the complete iteration of $i(V_W) = N_{i(W)}$ by $i(j_W(\vec{U})) \restriction (\kappa, j_W(\kappa))$. As a consequence $\vec{t} = \vec{s}^\frown\vec{r}_0$ where $\vec{r}_0$ is the sequence of sets of indiscernibles associated with $i'_{\kappa,j_W(\kappa)}$.

Let $\tilde{i} : N_{i(W)} \to \tilde{N}$ be the complete iteration of $N_{i(W)}$ by $j_{i(W)}(i(\vec{U})) \restriction (\kappa, j_{i(W)}(\kappa))$, and again, let $\vec{r}_1$ be the sequence of sets of indiscernibles associated with $\tilde{i}$. By Kaplan's theorem (Corollary 3.8) applied in $N$, $N[G_{\vec{s}}]_{W^*} = \tilde{N}[G_{\vec{s}} * G_{\vec{r}_1}]$ and $j_{W^*} = \tilde{i} \circ j_{i(W)}$.

We claim that $\tilde{i} = i'_{\kappa,j_W(\kappa)}$ and $\vec{r}_0 = \vec{r}_1$, which implies that $\tilde{N} = N'$, and

$$N[G_{\vec{s}}]_{W^*} = \tilde{N}[G_{\vec{s}} * G_{\vec{r}_1}] = N'[G_{\vec{s}} * G_{\vec{r}_0}] = j_W(N[G_{\vec{s}}]).$$

To prove the claim, note that $i(j_W(\vec{U})) = j_{i(W)}(i(\vec{U}))$ and therefore it suffices to show that $j_W(\kappa) = j_{i(W)}(\kappa)$. This is a consequence of the following computation:

$$j_{i(W)}(\kappa) = j_{i(W)}(i(\kappa)) = i(j_W(\kappa)) = j_W(\kappa).$$

The final equality requires some explanation. The point is that $i \restriction V_W$ is the complete iteration of $V_W$ by $\vec{U}$ and $j_W(\kappa)$ is a $V_W$-inaccessible cardinal greater than $\sup(\Delta)$.

For the fact that $N[G_{\vec{s}}]_{W^*} = j_W(N[G_{\vec{s}}])$ it follows that $j_{W^*}$ and $j_W \restriction N[G_{\vec{s}}]$ have the same domain and target models. Also,

$$j_{W^*} \circ i = \tilde{i} \circ j_{i(W)} \circ i = \tilde{i} \circ i \circ j_W = i' \circ j_W = j_W(i) \circ j_W = j_W \circ i$$

Since $j_{W^*}$ and $j_W$ have critical point $\kappa$, the two embeddings agree on $i[V]\cup\kappa$. Since $N = \mathrm{Hull}(i[V]\cup\kappa)$, $j_{W^*}\restriction N = j_W\restriction N$. Moreover,

$$j_{W^*}(G_{\vec{s}}) = G_{\vec{s}^\frown\vec{r}_1} = G_{\vec{s}^\frown\vec{r}_0} = j_W(G_{\vec{s}}).$$

Hence $j_{W^*} = j_W\restriction N[G_{\vec{s}}]$.

Finally, $k$ is the unique elementary embedding $\ell : N_{i(W)} \to j_{W^*}(N)$ such that $\ell(\kappa) = \kappa$ and $\ell\circ j_{i(W)} = j_{W^*}\restriction N$. By the above computations, $\tilde{i}$ satisfies these and therefore $k = \tilde{i}$. □

**Corollary 3.10.** *Let $\vec{U}$ be a sequence of normal measures on a discrete set $\Delta$ with $\sup(\Delta) = \kappa$. Let $i = i_{0,\theta} : V \to N$ be the complete iteration of $V$ by $\vec{U}$, and $\vec{s} = \langle s^\theta_\alpha \mid \alpha\in i(\Delta)\rangle$ be the sequence of sets of indiscernibles, and let $G = G_{\vec{s}}$. Suppose $\lambda\notin\Delta$ be a measurable cardinal and $W$ be a normal ultrafilter on $\lambda$. Let $W^*$ be the $N[G_{\vec{s}}]$-ultrafilter generated by $i(W)$. Then $j_{W^*} = j_W\restriction N[G_{\vec{s}}]$.*

*Proof.* Decompose $\mathbb{P}_\kappa$ to $\mathbb{P}_\lambda * \mathbb{P}_{(\lambda,\kappa)}$. Let $\Delta_0 = \Delta\cap\lambda$ and let $i^0 : V\to N^0$ be the complete iteration of $V$ by $\vec{U}\restriction\Delta_0$. By Theorem 2.10, $i^0(W)$ generates an $N^0[G_{\vec{s}}\cap\mathbb{P}_\lambda]$-ultrafilter $W^*$. By the previous lemma,

$$j^{N^0[G_{\vec{s}}\cap\mathbb{P}_\lambda]}_{W^*} = j_W\restriction N^0[G_{\vec{s}}\cap\mathbb{P}_\lambda].$$

Note that $i^0(W) = i(W)$ and that $W^*$ is the $N[G_{\vec{s}}]$-ultrafilter generated by $i(W)$. We claim that

$$j^{N[G_{\vec{s}}]}_{W^*} = j^{N^0[G_{\vec{s}}\cap\mathbb{P}_\lambda]}_{W^*}\restriction N[G_{\vec{s}}]$$

which will prove the corollary. The claim follows from the fact that in $N^0[G_{\vec{s}}\cap\mathbb{P}_\lambda]$, $N[G_{\vec{s}}]$ is an inner model which is closed under $\lambda$-sequences. This is proved by applying Lemma 3.4 inside $N^0[G_{\vec{s}}\cap\mathbb{P}_\lambda]$ to the sequence of measures generated by $i^0(\vec{U}\restriction(\lambda,\kappa))$. □

3.3. **Normal measures and the complete iteration.** In our classification of lifts of ultrafilters under the Magidor iteration we will have to classify the possible extensions of the point-wise images of a normal ultrafilter under the complete iteration. The next lemmas will be used for that purpose.

**Lemma 3.11.** *If $i : M\to N$ is an elementary embedding, $U\in M$ is a normal ultrafilter (p-point is enough) on $\delta$, and $N = \mathrm{Hull}^N(i[M]\cup i(\delta))$, then $F\cup i[U]$ generates $i(U)$ where $F$ denotes the tail filter on $i(\delta)$.*

*Proof.* Let $X\in i(U)$. Since $N = \mathrm{Hull}^N(i[M]\cup i(\delta))$, there is $f\in M$ and $\eta < i(\delta)$ such that $X = i(f)(\eta)$. Changing $f$ if needed, we may assume that $f : \delta\to U$. Let $A^* = \Delta_{\alpha<\delta} f(\alpha)$, then by the normality assumption $A^*\in U$. It follows that $i(A^*)\setminus\eta+1 \subseteq i(f)(\eta) = X$, and therefore $X$ is in the filter generated by $F\cup i[U]$. □

**Lemma 3.12.** *Suppose $i : M\to N$ is the complete iteration of $\vec{U}$, $\delta = \min(\mathrm{dom}(\vec{U}))$, and $\eta\le i(\delta)$. Let $X = \mathrm{Hull}^N(i[M]\cup\eta)$, let $k : \bar{N}\to N$ be* *the inverse of the transitive collapse, and let* $\bar{i} = k^{-1} \circ i$*. Let* $n \leq \omega$ *be least such that* $\eta \leq i_{0n}(\delta)$*. Then* $\bar{i} = i_{0n}$ *and* $k = i_{n\infty}$*.*

*Proof.* Let $M_n$ be the (transitive) $n^{\text{th}}$-iterate of $M$ in $i$, namely $i_{n,\infty} : M_n \to M$, and $i_{0,n} : M \to M_n$. We claim that $X = i_{n,\infty}[M_n]$, from which it follows from uniqueness that $i_{n,\infty} = k$ and therefore $\bar{i} = k^{-1} \circ i = i_{0,n}$. Indeed, any $x \in X$ has the form $i(f)(\xi)$ for some $\xi < \eta \leq i_{0,n}(\delta)$. Each such ordinal can be represented using $\delta, i_{0,1}(\delta), ..., i_{0,n-1}(\delta)$ and therefore we may assume that $x = i(f)(\delta, i_{0,1}(\delta), ..., i_{0,n-1}(\delta))$ since the critical point of $i_{n,\infty}$ is $i_{0,n}(\delta)$, we have that

$$x = i_{n,\infty}(i_{0,n}(f)(\delta, i_{0,1}(\delta), ..., i_{0,n-1}(\delta))) \in \text{rng}(i_{n,\infty})$$

The other inclusion is similar. □

**Lemma 3.13.** *Suppose* $i : M \to N$ *is an elementary embedding* $\delta$ *is a regular cardinal, and* $\delta \leq \eta < i(\delta)$ *is such that* $\eta \in i(C)$ *for every closed unbounded set* $C \subseteq \delta$ *in* $M$*. Let* $X = \text{Hull}^N(i[M] \cup \eta)$*, let* $k : \bar{N} \to N$ *be the inverse of the transitive collapse, and let* $\bar{i} = k^{-1} \circ i$*. Then* $\bar{i}(\delta) = \eta$*.*

*Proof.* Since $k^{-1}$ is just the transitive collapse, it suffices to prove that there are no ordinals $\alpha \in X$ between $\eta$ and $i(\delta)$. Let $\alpha \in X$ be below $i(\delta)$. Then $\alpha = i(f)(\xi)$ for some $\xi < \eta$. We may assume that $f : \delta \to \delta$. Let $C_f \subseteq \delta$ be the club of closure points of $f$. Then the assumption of the Lemma, $\eta \in i(C_f)$, namely, $\eta$ is a closure point of $i(f)$. Since $\xi < \eta$, $\alpha = i(f)(\xi) < \eta$. Hence no ordinal in $X$ is between $\eta$ and $i(\delta)$. □

**Lemma 3.14.** *Suppose* $M$ *is a transitive model of set theory,* $i : M \to N$ *is the complete iteration of* $\vec{U}$ *in* $M$*,* $U$ *is a normal* $M$*-ultrafilter on* $\delta$*, where* $\delta$ *is the minimal element in* $\text{dom}(\vec{U})$*, and* $\tilde{U}$ *is a fine* $N$*-ultrafilter on an ordinal* $\eta \leq i(\delta)$ *extending* $\{i(A) \cap \eta : A \in U\}$*. Then one of the following holds:*

- $\{i_{0n}(\delta)\} \in \tilde{U}$ *for some* $n < \omega$*,*
- $\tilde{U} = i_{0n}(U)$ *for some* $n \leq \omega$*.*

*Proof.* If $\eta = \nu + 1$ is a successor ordinal, then since $\tilde{U}$ is fine, $\{\nu\} \in \tilde{U}$. In that case, for every $A \in U$, $\nu \in i(A)$. Note that since the critical point of $i_{\omega,\infty}$ is greater than $i_{0,\omega}(\delta)$, $i_{0,\omega}(A) = i(A)$. Since $i_{0,\omega}$ is the $\omega^{\text{th}}$ iterate of the normal measure $\vec{U}_\delta$, the only seeds for a normal ultrafilter in $i_{0,\omega}$ are the $i_{0,n}(\delta)$'s, hence $\nu = i_{0n}(\delta)$ for some $n < \omega$.

Now assume that $\eta$ is a limit ordinal. Since $\tilde{U}$ is fine, every set in $\tilde{U}$ is unbounded in $\eta$. It follows that for every closed unbounded set $C \subseteq \delta$ in $M$, $i(C) \cap \eta$ is unbounded in $\eta$, and therefore if $\eta < i(\delta)$, then $\eta \in i(C)$. Let $X = \text{Hull}^N(i[M] \cup \eta)$, let $k : \bar{N} \to N$ be the inverse of the transitive collapse of $X$, and let $\bar{i} = k^{-1} \circ i$.

Note that $\bar{i} : M \to \bar{N}$, $\bar{N} = \text{Hull}^{\bar{N}}(\bar{i}[M] \cup \eta)$, and $\bar{i}(\delta) = \eta$ by Lemma 3.13[3]. Lemma 3.11 implies that $F \cup \bar{i}[U]$ generates $\bar{i}(U)$, where $F$ denotes the tail filter on $\eta$. Since $\tilde{U}$ is fine, $F \subseteq \tilde{U}$, and $\bar{i}[U] = \{i(A) \cap \eta : A \in U\} \subseteq \tilde{U}$. Since $F \cup \bar{i}[U]$ generates $\bar{i}(U)$, it follows that $\bar{i}(U) \subseteq \tilde{U}$.

By Lemma 3.12, there is some $n \leq \omega$ such that $\bar{i} = i_{0n}$ and $k = i_{n\infty}$. Since $k = i_{n\infty}$ and $\eta \leq \text{crit}(k)$, we have $P(\eta) \cap \bar{N} = P(\eta) \cap N$, and therefore using the maximality of ultrafilters and the fact that $\bar{i}(U) \subseteq \tilde{U}$, we obtain $\bar{i}(U) = \tilde{U}$. Since $\bar{i} = i_{0n}$, we have $\tilde{U} = \bar{i}(U) = i_{0n}(U)$, which proves the lemma. □

## 4. Some ultrafilters in the discrete Magidor extension

Throughout this section, we fix a measurable cardinal $\kappa$, a discrete set of measurable cardinals $\Delta \subseteq \kappa$, a sequence $\vec{U} = \langle U_\delta : \delta \in \Delta \rangle$ of normal ultrafilters $U_\delta$ on $\delta$, and a $V$-generic filter $G$ on the discrete Magidor iteration $\mathbb{P}_\kappa$ associated with $\vec{U}$. This section is devoted to showing that Kaplan's theorem (Corollary 3.8) does not generalize to arbitrary $\kappa$-complete ultrafilters on $\kappa$.

### 4.1. An ultrafilter with infinitely many extensions.

In this subsection, we exhibit a $\kappa$-complete ultrafilter $W$ on $\kappa \times \kappa$ that, in $V[G]$, has infinitely many distinct extensions to $\kappa$-complete ultrafilters.

In general, Corollary 2.3 sets up an equivalent condition for an embedding $k : V[G] \to M$ being an ultrapower embedding by some $V[G]$-ultrafilter $\tilde{W}$. The ultrafilter $\tilde{W}$ is uniquely determined by the following three ingredients:

- $j_{\tilde{W}}^{V[G]} \restriction V$, the *restricted ultrapower embedding*
- $j_{\tilde{W}}(G)$, the *image generic*
- $[\text{id}]_{\tilde{W}}$, the *seed*

The first two ingredients determine $j_{\tilde{W}}^{V[G]}$ and given that $[\text{id}]_{\tilde{W}} = a$, we can recover $\tilde{W}$ as the ultrafilter derived from $j_{\tilde{W}}^{V[G]}$ and $a$.

Note that an arbitrary list of ingredients need not form a recipe for cooking up a genuine $V[G]$-ultrafilter. Suppose one is given an elementary embedding $j : V \to M$, an $M$-generic filter $H$ on $j(\mathbb{P}_\kappa)$ and a point $a \in M$. When is there a lift $\tilde{W}$ of $W$ whose restricted ultrapower is $j$, image generic is $H$, and seed is $a$? It is easy to see this is the case if and only if the following hold:

- $j[G] \subseteq H$.
- $M[H] = \text{Hull}^{M[H]}(j[V] \cup \{H, a\})$.
- $W$ is the ultrafilter derived from $j$ using $a$.

[3]Although in Lemma 3.13 we are assuming that $\eta < i(\delta)$, the conclusion $\eta = \bar{i}(\delta)$ is true also when $\eta = i(\delta)$. To see this, note that in this case $i(\delta) + 1 \subseteq X$ and therefore $\bar{i}(\delta) = k^{-1}(i(\delta)) = i(\delta) = \eta$.

Let us turn to the description of $W$. We start with any normal ultrafilter $D$ on $\kappa$. For each $\alpha < \kappa$, let

$$\delta(\alpha) = \min(\Delta \setminus \alpha)$$

and let

$$W = \sum_D U_{\delta(\alpha)}$$

In other words, a set $A \subseteq \kappa \times \kappa$ belongs to $W$ if and only if for $D$-almost all $\alpha$, for $U_{\delta(\alpha)}$-almost all $\beta$, $(\alpha, \beta) \in A$. The general analysis of sums of ultrafilters yields the following lemma:

**Lemma 4.1.** *Let* $\delta = \min(j_D(\Delta) \setminus \kappa)$ *and let* $U = j_D(\vec{U})_\delta$. *Then* $V_W = (V_D)_U$, $j_W = j_U \circ j_D$, *and* $[\mathrm{id}]_W = (\kappa, \delta)$.

A $V[G]$-ultrafilter $\tilde{W}$ given by the three ingredients extends $W$ if there is an elementary embedding $k : V_W \to j_{\tilde{W}}(V)$ such that the restricted ultrapower $j^{V[G]}_{\tilde{W}} \restriction V$ is equal to $k \circ j_W$ and $k([id]_W)$ is equal to $[id]_{\tilde{W}}$.

The first lift is the canonical extension $W^*$ (see Definition 3.6), whose three ingredients are $i^W \circ j_W$, $G \times G_{\vec{s}^W}$ and $(\kappa, \delta)$. Here $i^W$ is the complete iteration of $V_W$ by $j_W(\vec{U}) \restriction (\kappa, j_W(\kappa))$. (Note that the critical point of $i^W$ is above $\delta$.)

To define a non-canonical lift $W$, we will absorb $j_U$ into the complete iteration $i^D$. Formally, let $i^D : V_D \to N$ be the complete iteration of $V_D$ via $j_D(\vec{U}) \restriction (\kappa, j_D(\kappa))$ and let $\vec{s}^{\,D} = \langle s^D_\alpha : \alpha \in \Delta^N \setminus \kappa \rangle$ be the associated sets of indiscernibles. By Proposition 3.7, letting $D^*$ be the canonical extension of $D$, $j^{V[G]}_{D^*} \restriction V = i^D \circ j_D$ and $j^{V[G]}_{D^*}(G) = G \times G_{\vec{s}^D}$.

Since $U$ is the first ultrafilter used in the complete iteration of $j_D(\vec{U}) \restriction (\kappa, j_D(\kappa))$, it is easy to see that $N$ is also the final model of the complete iteration of $V_W = (V_D)_U$ via $j_W(\vec{U}) \restriction (\kappa, j_W(\kappa))$. Moreover, letting $i^W : V_W \to N$ denote the embedding associated with the complete iteration, we have

$$i^W \circ j_W = i^W \circ j_U \circ j_D = i^D \circ j_D$$

Similarly, letting $\vec{s}^{\,W} = \langle s^W_\alpha : \Delta^N \setminus \kappa \rangle$ denote the associated sequence of sets of indiscernibles and $\delta^* = i^D(\delta)$, we have

$$s^D_\alpha = \begin{cases} s^W_\alpha & \text{if } \alpha \neq \delta^* \\ s^W_{\delta^*} \cup \{\delta\} & \text{if } \alpha = \delta^* \end{cases}$$

It follows that $j_{D^*} \neq j_{W^*}$ since

$$j_{W^*}(G) = G \times G_{\vec{s}^W} \neq G \times G_{\vec{s}^D} = j_{D^*}(G)$$

yet

$$V[G]_{W^*} = N[G \times G_{\vec{s}^W}] = N[G \times G_{\vec{s}^D}] = V[G]_{D*}.$$

Now let $W'$ be the $V[G]$-ultrafilter on $\kappa \times \kappa$ derived from $j_{D^*}$ using $(\kappa, \delta)$. Then by Corollary 2.3 $j_{W'} = j_{D^*}$. To see that $W'$ lifts $W$, we note that $j_{D^*} \restriction V = i^D \circ j_W = i^W \circ j_W$ and again $i^W(\kappa, \delta) = (\kappa, \delta)$. So the three ingredients of $W'$ are $i^W \circ j_W, G \times G_{\vec{s}^D}$, and $(\kappa, \delta)$. As we noted above, $j_{D^*}(G) \neq j_{W^*}(G)$ and it follows that $W' \neq W^*$.

Thus we have constructed two distinct $V[G]$-ultrafilters extending $W$: the canonical one, and another that is Rudin-Keisler equivalent to $D^*$. Are these the only extensions? As the title of this subsection suggests, the answer is no and there are infinitely many more, falling into two countably infinite families: the first generalizing $W'$ and the second generalizing $W^*$. We begin with the generalizations of $W'$, which are a bit easier to describe as they are all Rudin-Keisler equivalent to $D^*$. Let $\langle \delta_n \rangle_{n<\omega}$ be the increasing enumeration of $s^D_{\delta^*}$, the set of indiscernibles associated with $\delta^*$ in the complete iteration of $V_D$.

**Definition 4.2.** Let $W^1_n$ be the $V[G]$-ultrafilter derived from $j^{V[G]}_{D^*}$ using $(\kappa, \delta_n)$.

The reason this forms an extension of $W$ is that we can represent $i^W \circ j_W$ differently. It is a well-known fact that $j_{U^{n+1}} = j_{j_{U^n}(U)} \circ j_{U^n} = (j_{U^n} \restriction V_U) \circ j_U$. Therefore

$$\begin{aligned} i^W \circ j_W &= i^D \circ j_D \\ &= i^D_{n+1,\theta} \circ j_{U^{n+1}} \circ j_D \\ &= i^D_{n+1,\theta} \circ (j_{U^n} \restriction V_U) \circ j_U \circ j_D \\ &= i^D_{n+1,\theta} \circ (j_{U^n} \restriction V_U) \circ j_W \end{aligned}$$

Once again it is not hard to see that Corollary 2.3 can be applied here to conclude that for every $n < \omega$, $j^{V[G]}_{W^1_n} = j^{V[G]}_{D^*}$ and $[\text{id}]_{W^1_n} = (\kappa, \delta_n)$. In other words, the three ingredients that determine $W^1_n$ are $i^D_{n+1,\theta} \circ j_{U^n} \restriction M_U \circ j_W$, $G \times G_{\vec{s}^D}$, and $(\kappa, \delta_n)$. Note that $i^D_{n+1,\theta}(j_{U^n}(\kappa, \delta)) = (\kappa, j_{U^n}(\delta)) = (\kappa, \delta_n)$ and therefore $W^1_n$ lifts $W$. All the $W^1_n$'s are distinct as they are derived from the same embedding (i.e., they are Rudin-Keisler equivalent) using different seeds.

We now turn to the second family of extensions of $W$: the generalizations of $W^*$, which will be denoted by $W^0_n$. We specify the ultrafilter $W^0_n$ by listing the three ingredients first. As in the case of $W^1_n$, the restricted ultrapower embedding associated with $W^0_n$ is $i^D \circ j_D$ and the seed is $(\kappa, \delta_n)$. The difference is in the image generic: we will have $j_{W^0_n}(G) \neq G \times G_{\vec{s}^D}$. Instead,

$$j_{W^0_n}(G) = G \times G_{\vec{s}^n}$$

where $\vec{s}^n$ is the sequence of sets of indiscernibles obtained from $\vec{s}^D$ by removing the ordinal $\delta_n$ from $s^D_{\delta^*}$; that is, $s^n_{\delta^*} = s^D_{\delta^*} \setminus \{\delta_n\}$ and for $\alpha > \delta^*$, $s^n_\alpha = s^D_\alpha$.

It is not entirely obvious that there is an extension of $W$ to a $V[G]$-ultrafilter that has this restricted ultrapower embedding, image generic, and seed. To show that $W^0_n$ exists, we need to prove that there is an elementary embedding $\ell : V[G] \to N[G \times G_{\vec{s}^n}]$ extending $i^W \circ j_W$ such that $\ell(G) = G \times G_{\vec{s}^n}$ and

$$N[G \times G_{\vec{s}^n}] = \mathrm{Hull}^{N[G \times G_{\vec{s}^n}]}(\ell\big[V[G]\big] \cup \{(\kappa, \delta_n)\}) \tag{1}$$

Then by Corollary 2.3, $\ell = j_{W_n^0}$ where $W_n^0$ is the $V[G]$-ultrafilter on $\kappa \times \kappa$ derived from $\ell$ using $(\kappa, \delta_n)$.

To show that $\ell$ exists, let $U_n$ denote the $n$-th iterate $j_{U^n}(U)$ of $U$, and note that $U_n$ is an $N[G_{\vec{s}^D}]$-ultrafilter on $\delta_n$, although $U_n$ is not an element of $N[G_{\vec{s}^D}]$. By Lévy-Solovay, $U_n$ generates an $N[G \times G_{\vec{s}^D}]$-ultrafilter $U_n^*$. We will set

$$\ell = j_{U_n^*} \circ j_{D^*} \tag{2}$$

By Proposition 3.5 applied in $V_D$, $j_{U_n}^{N[G_{\vec{s}^D}]} \circ i^D = i^D$, and therefore $\ell$ extends $i^D \circ j_D$; moreover, $j_{U_n}(G_{\vec{s}}) = G_{\vec{s}^n}$, which implies that $\ell(G) = G \times G_{\vec{s}^n}$. This verifies that $\ell$ has the correct restricted embedding and image generic.

Finally, we verify (1). Let $H = \mathrm{Hull}^{N[G \times G_{\vec{s}}]}(\ell\big[V[G]\big] \cup \{\kappa, \delta_n\})$. Since $N[G \times G_{\vec{s}}] = V[G]_{D^*}$, we have

$$N[G \times G_{\vec{s}}] = \mathrm{Hull}^{N[G \times G_{\vec{s}}]}(j_{D^*}\big[V[G]\big] \cup \{\kappa\})$$

To show $N[G \times G_{\vec{s}}] = H$, it therefore suffices to show that $j_{D^*}\big[V[G]\big]$ is contained in $H$. For this, since $j_{D^*} \restriction V = \ell \restriction V$, it is enough to show that $j_{D^*}(G) \in H$. But $j_{D^*}(G) = G \times G_{\vec{s}}$ is definable from $\ell(G) = G \times G_{\vec{s}^n}$ and $\delta_n$ since, roughly speaking, $G_{\vec{s}} = G_{\vec{s}^n} \cup \{\delta_n\}$. Since $j_{D^*}(G)$ is definable from parameters in $H$, it belongs to $H$.

*Remark* 4.3. Note that we defined the extension $W_n^0$ essentially by removing a single ordinal from $\vec{s}^D$, to obtain $\vec{s}^n$. One might be tempted to define other similar extensions of $W$, instead using sequences $\vec{t}$ obtained by removing more elements of $\vec{s}^n$. But in fact, using such a sequence $\vec{t}$ in our specification of the three ingredients that constitute a lift of $W$ would be fallacious, because these ingredients do not correspond to any ultrafilter in $V[G]$. The reason is that removing any element of $\vec{s}^D$ other than $\delta_n$ makes Equation (1) above *false.*

We note the following theorem, which follows from the foregoing analysis of $W_1^0$:

**Theorem 4.4.** *In $V[G]$, for any normal ultrafilter $F$ on $\kappa$, there is a nontrivial elementary embedding from $V[G]_F$ to itself.*

*Proof.* By Kaplan's theorem (Corollary 3.8), $F = D^*$ for some normal ultrafilter $D$ of $V$. As above, let $U = j_D(\vec{U})_{\delta^*}$ and let $U^*$ be the $V[G]_{D^*}$-ultrafilter generated by $U$. Then $(V[G]_{D^*})_U = V[G]_{D^*}$, and so $j_U : V[G]_{D^*} \to V[G]_{D^*}$ is a nontrivial elementary embedding. □

4.2. **Classifying the extensions of $W$.** In this section, we prove the special case of our classification of ultrafilters in $V[G]$ for the ultrafilter $W = \sum_D U_{\delta(\alpha)}$.

**Theorem 4.5.** *If $\bar{W}$ is a countably complete $V[G]$-ultrafilter extending $W$, then $\bar{W} = W^i_n$ for some $i \in \{0,1\}$ and $n < \omega$.*

This proof contains most of the key ideas of the classification and avoids some notational difficulties involved in propagating the result to arbitrary ultrafilters.

*Proof.* Let $\bar{W}$ be an extension of $W$ to a $V[G]$-ultrafilter. Since $\bar{W}$ extends $W$, $(\pi_0)_*(\bar{W})$ extends $D$ where $\pi_0 : \kappa \times \kappa \to \kappa$ denotes the projection to the first coordinate. By Kaplan's theorem (Corollary 3.8), it follows that $(\pi_0)_*(\bar{W})$ must be equal to $D^*$. Therefore there is an elementary embedding $k : V[G]_{D^*} \to V[G]_{\bar{W}}$ such that $k \circ j_{D^*} = j_{\bar{W}}$ and $k(\kappa) = \kappa$.

Note that $[\mathrm{id}]_{\bar{W}} = (\kappa, \bar{\delta})$ for some ordinal $\bar{\delta} > \kappa$. Let $\eta$ be the least ordinal such that $k(\eta) > \bar{\delta}$ and let $\bar{U}$ denote the $V[G]_{D^*}$-ultrafilter on $\eta$ derived from $k$ using $\bar{\delta}$. By Corollary 2.3, $k = j_{\bar{U}}$ and $[id]_{\bar{U}} = \bar{\delta}$.

**Claim 4.6.** *For some $n < \omega$, $\bar{\delta} = \delta_n$ and either $\bar{U} = U^*_n$ or $\{\delta_n\} \in \bar{U}$, so $\bar{U}$ is principal.*

Here $U^*_n$ is defined as in the paragraph preceding (2). Granting this claim, it is easy to see that either $\bar{W} = W^0_n$ or $\bar{W} = W^1_n$. Indeed if $\{\delta_n\} \in \bar{U}$ then $j_{\bar{U}}$ is the identity and $j_{\bar{W}} = j_{D^*}$. Then $\bar{W}$ is derived from $j_{D^*}$ using $(\kappa, \delta_n)$ which is by definition the ultrafilter $W^0_n$. If $\bar{U} = U^*_n$, then $\bar{W}$ is derived from $j_{U^*_n} \circ j_{D^*}$ using $(\kappa, \delta_n)$, which is by definition $W^0_n$.

To prove the claim, we analyze the $N$-ultrafilter $\tilde{U} = \bar{U} \cap N$. We will show that for some $n$, either $\tilde{U} = U_n$ or $\{\delta_n\} \in \tilde{U}$. The claim then follows since in either case, $\tilde{U}$ generates an $N[G \times G_{\vec{s}}]$-ultrafilter.

The analysis of $\tilde{U}$ is an application of Lemma 3.14 in the case $M = V_D$ and $i = i^D$, which implies that either $\tilde{U}$ has the desired form or else $\tilde{U} = i^D_{0\omega}(U)$. But the latter cannot occur, because $\tilde{U}$ extends to a countably complete $V[G]_{D^*}$-ultrafilter (namely, $\bar{U}$), whereas $i^D_{0\omega}(U)$ does not since $i^D_{0\omega}(\delta)$ has countable cofinality in $V[G]_{D^*}$. $\square_{\text{Theorem 4.5}}$

## 5. Classification of ultrafilters in the Magidor extension

5.1. **Extensions of an iterated sum of normal ultrafilters.** Let $M$ be a transitive model of set theory.

**Definition 5.1.** A *finite iteration of $M$* is a sequence $(D_m : m < n)$ such that for each $m < n$, $D_m$ is a $M_{D_0,\cdots,D_{m-1}}$-ultrafilter on an ordinal $\delta_m > \delta_{m-1}$.

We say that an iteration $(D_0, ..., D_{n-1})$ is *below* $\gamma$ if for each $m < n$, the ultrafilter $D_m$ lies on an ordinal less than $j_{D_0,...,D_{m-1}}(\gamma)$.

**Definition 5.2.** The *sum of* $(D_0, \dots, D_{n-1})$, denoted $\Sigma(D_0, \dots, D_{n-1})$, is the unique ultrafilter $W$ on $\kappa^n$ such that $j_W = j_{D_0,\dots,D_{n-1}}$ and $[\text{id}]_W = (\delta_0, \delta_1, \dots, \delta_n)$, where $\kappa$ is the least ordinal such that

$$(\delta_0, \dots, \delta_{n-1}) \in j_{D_0,\dots,D_{n-1}}(\kappa^n).$$

The purpose of this section is to define extensions of certain sums of normal ultrafilters to $V[G]$ where $G \subseteq \mathbb{P}_\kappa$ is $V$-generic. We will assume that for each $m < n$, $D_m \in V_{D_0,\dots,D_{m-1}}$, in which case we call the iteration *internal*. The following lemma explains why it is natural to consider such sums:

**Theorem 5.3** (UA). *Assume that there is no cardinal $\kappa$ with $o(\kappa) = 2^{2^\kappa}$. Then every countably complete ultrafilter is Rudin-Keisler equivalent to the sum of an internal iteration of normal measures.*

Fix an internal iteration $(D_0, ..., D_{n-1})$ of normal ultrafilters. Let $M_m = V_{D_0,\dots,D_{m-1}}$ and let $j_{m_0 m_1} : M_{m_0} \to M_{m_1}$ be the iterated ultrapower embedding $j_{D_{m_0}\cdots D_{m_1-1}}$. Let $d$ be the set of $m \leq n$ such that $\delta_m \in j_{0m}(\Delta)$, and let $d'$ be the set of $m \in d$ such that $D_m = j_{0m}(\vec{U})_{\delta_m}$.

Fix $u : d' \to \{0,1\}$ and $x : d \to \omega$. By recursion on $m \leq n$ we will define a $V[G]$-ultrafilter $\bar{W}_m = \Sigma(D_0, \dots, D_{m-1})^u_x$ extending $W_m = \Sigma(D_0, \dots, D_{m-1})$. We will also define an external iteration $(E_0, E_1, ..., E_{m-1})$ of $V_{W_m}$ below $\delta_m$ whose well-founded last model $P_m$ completely iterates into $j^{V[G]}_{\bar{W}_m}(V)$. More precisely,

- Let $e_m : V_W \to P_m$ be the iterated ultrapower by $(E_0, ..., E_{m-1})$.
- Let $i^{P_m} : P_m \to N_m$ be the complete iteration of $P_m$ by $e_m(j_{W_m}(\vec{U}))$ above $\kappa$.

Then $N_m = j_{\bar{W}_m}(V)$, $j_{\bar{W}_m} \restriction V = i^{P_m} \circ e_m \circ j_{W_m}$ and $i^{P_m} \circ e_m([id]_{W_m}) = [id]_{\bar{W}_m}$. This ensures, in particular, that the ultrafilter $\bar{W}_m$ lifts $W_m$.

Fix $m < n$ and assume that we have already defined $\bar{W}_m$ and the associated external iteration $(E_0, ..., E_{m-1})$. We will define $\bar{W}_{m+1}$ and $E_m$.

Let $i : V_W \to N_m$ be the composition $i^{P_m} \circ e_m$. We will define an $N_m$-ultrafilter $\tilde{D}_m$ extending

$$\{i(A) \cap \eta : A \in D_m\}$$

for some $\eta \leq i(\delta_m)$. The ultrafilter $\tilde{D}_m$ will generate a $V[G]_{\bar{W}_m}$-ultrafilter, which we denote by $D^{u,x}_m$, and we will set $\bar{W}_{m+1} = \Sigma(\bar{W}_m, D^{u,x}_m)$. Therefore in the end we will have $\bar{W}_n = \Sigma(D^{u,x}_0, \dots, D^{u,x}_{n-1})$.

If $m \notin d$, $\tilde{D}_m = i(D_m)$ an a $E_m$ the principal $P_m$-ultrafilter concentrated at $e_m(\delta_m)$.

Now suppose $m \in d$, in which case $\tilde{D}_m$ will depend on $u$ and $x$. Define $E_m$ to be the external ultrafilter $e_m(j_{W_m}(\vec{U})_{\delta_m})^{x(m)}$. For each $\beta$ less than the length of the complete iteration of $P_m$, let $\delta^\beta_m = i^{P_m}_{0\beta}(e_m(\delta_m))$ and $D^\beta_m = i^{P_m}_{0\beta}(e_m(D_m))$.

Let $\alpha$ be the first stage of the complete iteration of $P_m$ such that $\mathrm{crit}(i^{P_m}_{\alpha\alpha+1}) = \delta^{\alpha}_m$. If $m \in d'$ and $u(m) = 1$, let $\tilde{D}_m$ be the principal ultrafilter concentrated at $\delta^{\alpha+x(m)}_m$. Otherwise, let $\tilde{D}_m = D^{\alpha+x(m)}_m$.

We have $\{i(A) \cap \eta : A \in D_m\} \subseteq \tilde{D}_m$ where

$$\eta = \begin{cases} i(\delta_m) & \text{if } m \notin d \\ \delta^{\alpha+x(m)}_m + 1 & m \in d' \text{ and } u(m) = 1 \\ \delta^{\alpha+x(m)}_m & \text{otherwise} \end{cases}$$

**Claim 5.4.** *$\tilde{D}_m$ generates a $V[G]_{\bar{W}_m}$-ultrafilter $D^{u,x}_m$*

*Proof.* The proof is by cases. In the first case when $m \notin d$ we appeal to Theorem 2.10. In the second case, when $m \in d'$ and $u(m) = 1$, $\tilde{D}_m$ is principal and therefore trivially generates a $V[G]_{\bar{W}_m}$-ultrafilter. In the last case, $\tilde{D}_m$ is a $\gamma$-complete $N_m$-ultrafilter on $\gamma = \delta^{\alpha+x(m)}_m$, so it generates an ultrafilter by Lévy-Solovay. $\square$

Finally, define

$$\bar{W}_{m+1} = \Sigma(\bar{W}_m, D^{u,x}_m).$$

To complete the induction, we must prove the following claim:

**Claim 5.5.** *$N_{m+1} = j_{\bar{W}_{m+1}}(V)$, $j_{\bar{W}_{m+1}} \restriction V = i^{P_{m+1}} \circ e_{m+1} \circ j_{W_{m+1}}$, and $i^{P_{m+1}} \circ e_{m+1}([id]_{W_{m+1}}) = [id]_{\bar{W}_{m+1}}$.*

*Proof.* We consider the three cases. For the first case, assume $m \notin d$. Let $G_m$ be the $N_m$-generic filter given by the sequence of sets of indiscernibles associated with the complete iteration $i^{P_m}$. By the induction hypothesis, $V[G]_{\bar{W}_m} = N_m[G \times G_m]$. By definition, in this case, $\tilde{D}_m = i(D_m)$. Let $D^*_m$ be the $N_m[G_m]$-ultrafilter generated by $\tilde{D}_m$.

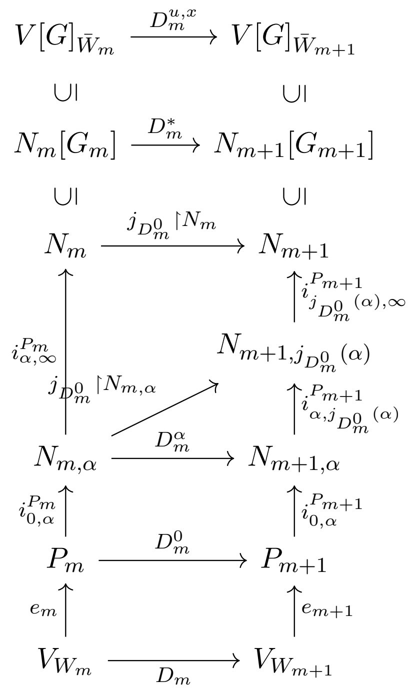


FIGURE 1. The case that $m \notin d$

Note that by Lévy-Solovay,

$$j_{D_m^{u,x}} \restriction N_m[G_m] = j_{D_m^*}^{N_m[G_m]}$$

The key point is that

$$j_{D_m^*}^{N_m[G_m]} \restriction N_m = j_{D_m^0} \restriction N_m \tag{3}$$

To see this, let $\alpha$ be the least ordinal such that $\mathrm{crit}(i^{P_m}_{\alpha,\alpha+1}) > e_m(\delta_m)$ and consider the model $N_{m,\alpha}[G_m \restriction \delta_m^0]$. Note that $G_m \restriction \delta_m^0$ is $N_{\alpha,m}$-generic for a forcing which has smaller cardinality than the critical point of the embedding $i^{P_m}_{\alpha,\infty}$. So by the Lévy-Solovay argument, $i^{P_m}_{\alpha,\infty}$ lifts to an embedding $i^*_{\alpha,\infty} : N_{m,\alpha}[G_m \restriction \delta_m^0] \to N_m[G_m \restriction \delta_m^0]$. It is easy to see that $i^*_{\alpha,\infty}$ is the complete iteration of $N_{m,\alpha}[G_m \restriction \delta_m^0]$ via the canonical lift of the sequence $i^{P_m}_{0\alpha}(e_m(\vec{U})) \restriction (\delta_m^0, \infty)$. Moreover $N_m[G_m]$ is the generic extension of the final model of this iteration by the filter obtained from the associated sequence of sets of indiscernibles. Therefore we can apply Lemma 3.4 in $N_{m,\alpha}[G_m \restriction \delta_m^0]$ to conclude that $N_m[G_m]$ is closed under $\delta_m^0$-sequences from $N_{m,\alpha}[G_m \restriction \delta_m^0]$.

Note that

$$j_{D_m^\alpha} \circ i^{P_m}_{0,\alpha} = i^{P_{m+1}}_{0,\alpha} \circ j_{D_m^0}. \tag{4}$$

This follows once we prove that $i^{P_{m+1}}_{0,\alpha} = i^{P_m}_{0,\alpha} \restriction P_{m+1}$. This is a routine induction on $\beta \leq \alpha$ using that $\text{crit}(j_{D^0_\alpha})$ is greater than all the measurable cardinals appearing in the iteration $i^{P_m}_{0,\alpha}$.

By Mitchell's lemma (Lemma 3.9) applied in $P_m$ with $W = D^0_m$, $j^{N_{m,\alpha}[G_m \restriction \delta^0_m]}_{D^*_m} = j_{D^0_m} \restriction N_{m,\alpha}[G_m \restriction \delta^0_m]$. Since $N_m[G_m]$ is closed under $\delta^0_m$-sequences from $N_{m,\alpha}[G_m \restriction \delta^0_m]$, this implies that

$$j^{N_m[G_m]}_{D^*_m} = j^{N_{m,\alpha}[G_m \restriction \delta^0_m]}_{D^*_m} \restriction N_m[G_m] = j_{D^0_m} \restriction N_m[G_m] \tag{5}$$

which proves (3).

By definition of $e_{m+1}$, $e_{m+1} = e_m \restriction V_{W_{m+1}}$ ensuring that

$$j_{D^0_m} \circ e_m = e_{m+1} \circ j_{D_m}.$$

Also, $i^{P_{m+1}} = j_{D^0_m}(i^{P_m})$. We have

$$j_{D^{u,x}_m} \circ i^{P_m} = j^{N[G_m]}_{D^*_m} \circ i^{P_m} = j_{D^0_m} \circ i^{P_m} = i^{P_{m+1}} \circ j^{P_m}_{D^0_m}$$

Moreover

$$j^{P_m}_{D^0_m} \circ e_m = e_{m+1} \circ j_{D_m}$$

Combining these equations, we get

$$\begin{aligned} i^{P_{m+1}} \circ e_{m+1} \circ j_W &= i^{P_{m+1}} \circ e_{m+1} \circ j_{D_m} \circ j_{W_m} \\ &= i^{P_{m+1}} \circ j^{P_m}_{D^0_m} \circ e_m \circ j_{W_m} \\ &= j_{D^{u,x}_m} \circ i^{P_m} \circ e_m \circ j_{W_m} \\ &= j_{D^{u,x}_m} \circ j_{\bar{W}_m} \restriction V \\ &= j_{\bar{W}_{m+1}} \restriction V \end{aligned}$$

To finish the case $m \notin d$, by the normality of $D^{u,x}_m$, $[id]_{D^{u,x}_m}$ is the ordinal $\beta$ over which $D^{u,x}_m$ is an ultrafilter. On the other hand, $i^{P_{m+1}}(e_{m+1}(\delta_m)) = i^{P_{m+1}}_{0,\alpha}(e_m(\delta_m)) = \beta$. Hence $i^{P_{m+1}}(e_{m+1}([id]_{W_{m+1}}) = [id]_{\bar{W}_{m+1}}$.

Next consider the case where $m \in d'$ and $u(m) = 1$. By definition $\tilde{D}_m$ is $p_{\delta^{\alpha+x(m)}_m}$, the principal ultrafilter concentrated at $\delta^{\alpha+x(m)}_m$. The following diagram commutes:

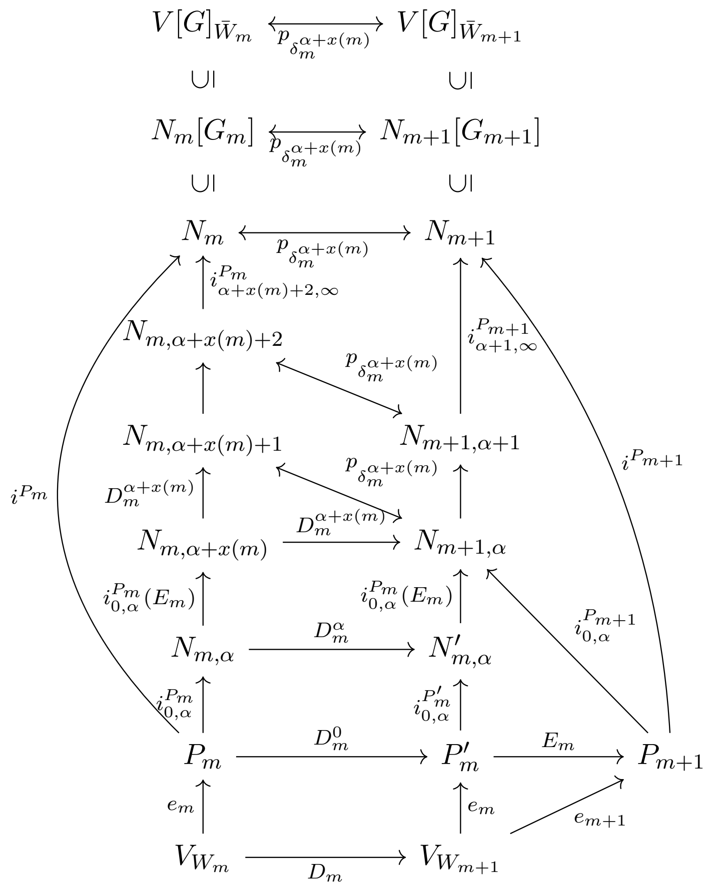


FIGURE 2. The case that $u(m) = 1$.

The commutativity of the second square from the bottom is proved as in Equation 4. The only other part of the diagram whose commutativity is not immediate is

$$i^{P_{m+1}}_{0,\alpha} \circ j_{E_m} = j_{i^{P_m}_{0,\alpha}(E_m)} \circ i^{P'_m}_{0,\alpha}$$

where $P'_m = e_m(V_{W_{m+1}})$ and $i^{P'_m}_{0,\alpha} : P'_m \to N'_{m,\alpha}$ is the $\alpha^{\text{th}}$ stage of the complete iteration of $P'_m$ by $e_m(j_{W_{m+1}}(\vec{U}))$ above $\kappa$.

$$\begin{array}{ccc} N'_{m,\alpha} & \xrightarrow{i^{P_m}_{0,\alpha}(E_m)} & N_{m+1,\alpha} \\ {\scriptstyle i^{P'_m}_{0,\alpha}}\uparrow & & \uparrow{\scriptstyle i^{P_{m+1}}_{0,\alpha}} \\ P'_m & \xrightarrow{E_m} & P_{m+1} \end{array}$$

This commutativity is true since

$$j^{N'_{m,\alpha}}_{i^{P_m}_{0,\alpha}(E_m)} = j_{E_m} \restriction N'_{m,\alpha} \text{ and } i^{P_{m+1}}_{0,\alpha} = j_{E_m}(i^{P'_m}_{0,\alpha}).$$

We include some details on how to show $j^{N'_{m,\alpha}}_{i^{P_m}_{0,\alpha}(E_m)} = j_{E_m} \restriction N'_{m,\alpha}$. For this, we use that $j^{N'_{m,\alpha}}_{i^{P_m}_{0,\alpha}(E_m)} = j^{N_{m,\alpha}}_{i^{P_m}_{0,\alpha}(E_m)} \restriction N'_{m,\alpha}$. Since $E_m \in P_m$, we can use Kunen's commuting ultrapowers lemma, as it appears in Woodin's [13, Lemma 3.30], to conclude that

$$j^{N_{m,\alpha}}_{i^{P_m}_{0,\alpha}(E_m)} = j^{P_m}_{E_m} \restriction N_{m,\alpha}.$$

(We apply Woodin's lemma in $P_m$ with $j = i^{P_m}_{0,\alpha}$ and $E = E_m$. Note that some of the generality of Woodin's lemma is not necessary here since $j$ is definable over $P_m$ rather than generic.)

Since $j_{\bar{W}_{m+1}} = j_{\bar{W}_m}$ then the commutativity of the diagram in Figure 2 can be used to deduce that

$$j_{\bar{W}_{m+1}} \restriction V = j_{\bar{W}_m} \restriction V = i^{P_m} \circ e_m \circ j_{W_m} = i^{P_{m+1}} \circ e_{m+1} \circ j_{W_{m+1}}.$$

In particular,

$$N_{m+1} = N_m = j_{\bar{W}_m}(V) = j_{\bar{W}_{m+1}}(V)$$

Finally, we show that $i^{P_{m+1}} \circ e_{m+1}([id]_{W_{m+1}}) = [id]_{\bar{W}_{m+1}}$. By the induction hypothesis it suffices to show that $i^{P_{m+1}}(e_{m+1}(\delta_m)) = \delta_m^{\alpha+x(m)}$:

$$i^{P_{m+1}}(e_{m+1}(\delta_m)) = i^{P_{m+1}}_{\alpha,\infty}(i^{P_{m+1}}_{0,\alpha}(e_{m+1}(\delta_m))) = i^{P_{m+1}}_{\alpha,\infty}(j_{i^{P_m}_{0,\alpha}(E_m)}(i^{P'_m}_{0,\alpha}(e_m(\delta_m)) =$$

$$= i^{P_{m+1}}_{\alpha,\infty}(j_{i^{P_m}_{0,\alpha}(E_m)}(i^{P_m}_{0,\alpha}(e_m(\delta_m)) = i^{P_{m+1}}_{\alpha,\infty}(\delta_m^{\alpha+x(m)}) = \delta_m^{\alpha+x(m)}.$$

Finally consider the case where where either $m \in d \setminus d'$ or $m \in d'$ but $u(m) = 0$. Let $\ell$ be the restriction to $N_m$ of the ultrapower embedding of $N_{m,\alpha+x(m)}$ by $D_m^{\alpha+x(m)}$. Note that the bottom part of the diagram in Figure 3 is identical to the bottom part of the diagram in Figure 2 and in particular it commutes. In fact the whole diagram commutes and the key to that is that the embedding $\ell$ is the restriction to $N_m$ of $j^{N_m[G_m]}_{D^*}$ where $D^*$ is the $N_m[G_m]$-ultrafilter generated by $D_m^{\alpha+x(m)}$. The justification for this is as in Equation 5. The commutativity of the rest of the diagram is a straightforward verification, and the remainder of the proof of the claim in this case is then identical to the previous part.

This completes the proof, but let us note here that in the case where $m \in d'$ we obtain

$$N_{m+1} = j_{\bar{W}_{m+1}}(V) = j_{\bar{W}_m}(V) = N_m.$$

This is because in this case $N_{m+1,\alpha} = N_{m,\alpha+x(m)+1}$. Moreover,

$$V[G]_{\bar{W}_{m+1}} = N_{m+1}[G \times G_{m+1}] = N_m[G \times G_m] = V[G]_{\bar{W}_m} \tag{6}$$

Indeed $G_m$ and $G_{m+1}$ differ by exactly one ordinal since they are given the sequences of sets of indiscernibles associated with essentially the same complete iterations (see also Proposition 3.5). □

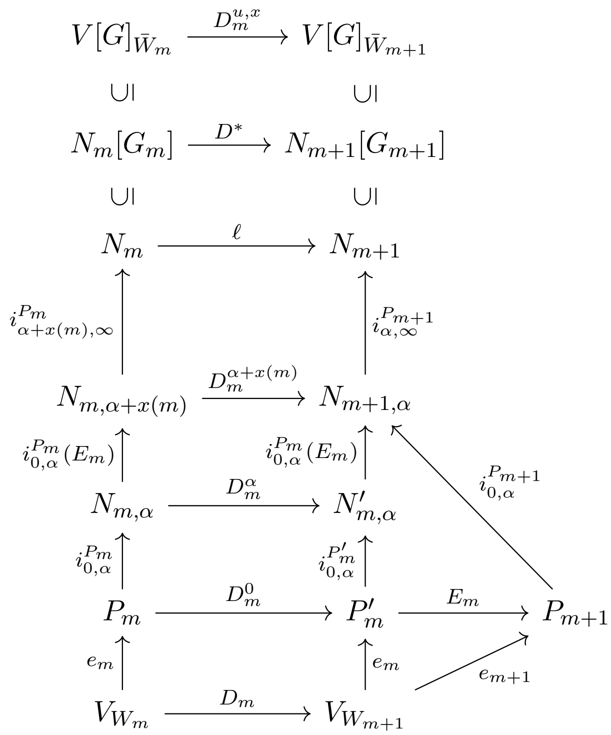


FIGURE 3. The case where $u(m) \neq 1$

**Lemma 5.6.** *Fix an internal iteration $(D_0, ..., D_{n-1})$ of normal ultrafilters, let $d$ and $d'$ be as in the paragraph following Theorem 5.3. Suppose that $d = d'$ and fix $u : d \to \{0, 1\}$ and $x : d \to \omega$. Let $\bar{W} = \sum(D_0, ..., D_{n-1})^u_x$. Then in $V[G]$ there is an internal iteration $(F_0, ..., F_{\ell-1})$ of normal ultrafilters such that $V[G]_{\bar{W}} = V[G]_{F_0,...,F_{\ell-1}}$.*

*Proof.* An easy induction using the definition of $\sum(D_0, ..., D_{n-1})^u_x$ in the case $d = d'$. Note that in the case that $m \notin d$, $D^{u,x}_m$ is an internal normal ultrafilter of $V[G]_{\bar{W}_m}$. If $m \in d$ then $m \in d'$ by our assumption and $V[G]_{\bar{W}_m} = V[G]_{\bar{W}_{m+1}}$ which follows by Equation 6 if $u(m) = 0$ or since $D^{u,x}_m$ is principal in the case that $u(m) = 1$. □

5.2. **Classifying the extensions of sums of normals.** In this section, we classify the extensions to $V[G]$ of sums of normal ultrafilters; i.e., ultrafilters of the form $\Sigma(D_0, ..., D_n)$. As expected, the proof is by induction on $n$. Recall that given a finite iteration $(D_0, ..., D_n)$, we define $d = d(D_0, ..., D_n)$ as the set of all $m \leq n$ such that $\delta_m \in j_{0,m}(\Delta)$ and $d' = d'(D_0, ..., D_n)$ as the set of $m \in d$ such that $D_m = j_{0,m}(\vec{U})_{\delta_m}$.

**Theorem 5.7.** *Let $\bar{W}$ be a countably complete $V[G]$-ultrafilter extending $\Sigma(D_0, ..., D_n)$. Then $\bar{W} = \Sigma(D_0, ..., D_n)^u_x$ for some $u : d' \to \{0, 1\}$ and $x : d \to \omega$.*

*Proof.* We follow a similar argument to the one in Theorem 4.5. Suppose inductively that we have classified the extensions of $W_m = \Sigma(D_0, ..., D_{m-1})$ and let us classify the extensions of $W_{m+1} = \Sigma(D_0, ..., D_m)$. Let $\bar{W}_{m+1}$ be an ultrafilter on $\kappa^{m+1}$ extending of $W_{m+1}$ to a $V[G]$-ultrafilter. Note that $[\mathrm{id}]_{\bar{W}_{m+1}} = (\bar{\delta}_0, ...\bar{\delta}_m)$ for some increasing sequence of ordinals. Therefore there is a factor map $k : V_{W_{m+1}} \to j_{\bar{W}_{m+1}}(V)$ such that $j_{\bar{W}_{m+1}} \restriction V = k \circ j_{W_{m+1}}$ and $k([id]_{W_{m+1}}) = [id]_{\bar{W}_{m+1}}$.

Since $\bar{W}_{m+1}$ extends $W_{m+1}$, $\pi_*(\bar{W}_{m+1})$ extends $W_m$ where $\pi : \kappa^{m+1} \to \kappa^m$ denotes the projection to the first $m$-coordinates. By the induction hypothesis, it follows that $\pi_*(\bar{W}_{m+1})$ must be equal to $\bar{W}_m = \Sigma(D_0, ..., D_{m-1})^{u'}_{x'}$ for some $u', x'$. Therefore there is an elementary embedding $\ell : V[G]_{\bar{W}_m} \to V[G]_{\bar{W}_{m+1}}$ such that $\ell \circ j_{\bar{W}_m} = j_{\bar{W}_{m+1}}$ and $\ell([id]_{\bar{W}_m}) = (\bar{\delta}_0, ..., \bar{\delta}_{m-1})$.

Let $\eta$ be the least ordinal such that $\ell(\eta) > \bar{\delta}_m$ and let $\bar{U}$ denote the $V[G]_{\bar{W}_m}$-ultrafilter on $\eta$ derived from $\ell$ using $\bar{\delta}_m$. By Corollary 2.3, $\ell = j_{\bar{U}}$ and $[id]_{\bar{U}} = \bar{\delta}_m$. Let $e_m, i^{P_m}, N_m$ be defined we defined before (after Theorem 5.3) for $\bar{W}_m = \Sigma(D_0, .., D_{m-1})^{u'}_{x'}$, and denote by $i_\alpha = i^{P_m}_{0,\alpha} \circ e_m : V_{W_n} \to N_m$ and let $i = i_{0,\infty} = i^{P_m} \circ e_m$. Let $\tilde{U} = \bar{U} \cap N_m$, then there is a factor map $k_{\tilde{U}} : (N_m)_{\tilde{U}} \to j_{\bar{W}_{m+1}}(V)$ such that $j_{\bar{U}} \restriction N_m = k_{\tilde{U}} \circ j_{\tilde{U}}$ and $k_{\tilde{U}}([id]_{\tilde{U}}) = \bar{\delta}_m$. Then the following diagram commutes:

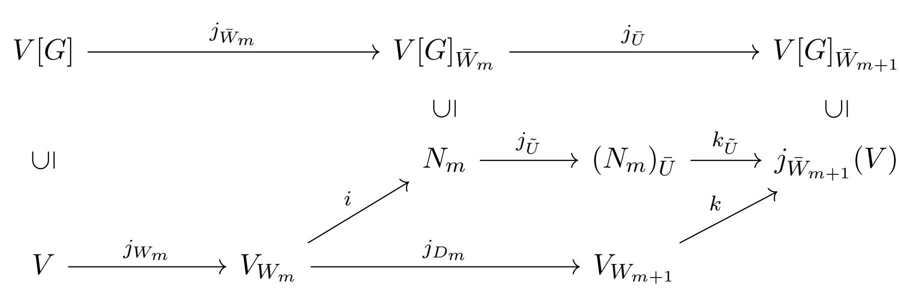

FIGURE 4. The decomposition of $j_{\bar{W}_{m+1}}$.

We claim that $i[D_m] \subseteq \tilde{U}$. To see this, let $X \in D_m$, then

$$\begin{aligned} [id]_{D_m} \in j_{D_m}(X) &\Rightarrow \bar{\delta}_m \in k(j_{D_m}(X)) \\ &\Rightarrow k_{\tilde{U}}([id]_{\tilde{U}}) \in k_{\tilde{U}}(j_{\bar{U}}(i(X))) \\ &\Rightarrow [id]_{\tilde{U}} \in j_{\bar{U}}(i(X)) \Rightarrow i(X) \in \tilde{U} \end{aligned}$$

Next we will prove that $\tilde{U}$ is equal to one of the following ultrafilters:

- $i(D_m)$.
- $p_{\delta_m^{\alpha+n}}$ for some $n < \omega$.

- $D_m^{\alpha+n}$ for some $n<\omega$.

Once we prove the above, it will follow that $\tilde{U}$ is the ultrafilter that was denoted by $\tilde{D}_m$ which by the previous part generates the ultrafilter $D_m^{u,x}$ in $V[G]_{\bar{W}_m}$. Hence $\bar{U}=D_m^{u,x}$ for an appropriate extension of $u',x'$ to $u,x$ determined by the value of $\tilde{U}$. This will end the proof as by definition,

$$\bar{W}_{m+1}=\Sigma(\bar{W}_m,\bar{U})=\Sigma(\bar{W}_m,D_m^{u,x})=\Sigma(D_0,...,D_m)_x^u$$

Let $\alpha$ be the first stage of the iteration such that the critical point of $i^{P_m}_{\alpha,\infty}$ is at least $i_\alpha(\delta_m)$. We will show that $i^{P_m}_{\alpha,\infty}[i_\alpha(D_m)]\subseteq\tilde{U}$. Note that every generator of $i_\alpha$ is less than $i_\alpha(\delta_m)$; this follows from our choice of $\alpha$ and the fact that $e_m$ is an iteration of ultrafilters on cardinals below $\delta_m$. Similarly, $i_\alpha$ is continuous at $\delta_m$ and therefore $i_\alpha[D_m]$ generates $i_\alpha[D_m]\cup F$, where $F$ is the tail filter on $i_\alpha(\delta_m)$. Applying Lemma 3.11 we conclude that $i_\alpha[D_m]$ generates $i_\alpha(D_m)$. Since $i[D_m]\subseteq\tilde{U}$ it follows that $i^{P_m}_{\alpha,\infty}[i_\alpha(D_m)]\subseteq\tilde{U}$.

We first consider the case where $m\notin d$, meaning $\delta_m\notin j_{0,m}(\Delta)$. Hence $\mathrm{crit}(i^{P_m}_{\alpha,\infty})>i_\alpha(\delta_m)$. It follows that $\tilde{U}=i_\alpha(D_m)=i(D_m)$.

Now suppose that $m\in d$, namely $\delta_m\in j_{0,m}(\Delta)$. The analysis of $\tilde{U}$ is an application of Lemma 3.14 in the case $M=N^m_\alpha$, $i=i^{P_m}_{\alpha,\infty}$ and the ultrafilter $i_\alpha(D_m)$ which is a normal ultrafilter on $i_\alpha(\delta_m)$, the minimal ordinal in the remaining part of the complete iteration $i^{P_m}_{\alpha,\infty}$ as computed in $N_{m,\alpha}$. We conclude that either $\tilde{U}$ has the desired form or else $\tilde{U}=i_{\alpha+\omega}(D_m)$. But the latter cannot occur, because $\tilde{U}$ extends to a countably complete $V[G]_{\bar{W}_m}$-ultrafilter (namely, $\bar{U}$), whereas $i_{\alpha+\omega}(D_m)$ does not since $i_{\alpha+\omega}(\delta_m)$ has countable cofinality in $V[G]_{\bar{W}_m}$. $\square_{\text{Theorem 5.7}}$

## 6. Applications

Our first application resolves the problem of whether Weak UA is equivalent to UA (see [5, Question 9.2.4]).

**Lemma 6.1.** *Assume that the Mitchell order is linear on normal ultrafilters, and that for every $\sigma$-complete ultrafilter $U$ there is an internal iteration of normal ultrafilters $(D_0,...,D_{n-1})$ such that $V_U=V_{D_0,...,D_{n-1}}$. Then Weak UA holds.*

*Proof.* Granting the linearity of the Mitchell order on normal measures, [5, Prop. 2.3.13] states that the Ultrapower Axiom for normal ultrafilters holds. It is then not hard to show that the Ultrapower Axiom holds for internal (finite) iterations of normal ultrafilters (see for example the proof [5, Prop. 8.3.43]). By the second assumption of the lemma, this suffices to compare (without commutativity) every two ultrapowers of $\sigma$-complete ultrafilters. $\square$

**Lemma 6.2.** *Suppose that the Mitchell order is linear in $V$ and that each normal ultrafilter of $V[G]$ is generated by an ultrafilter of $V$. Then the Mitchell order is linear in $V[G]$.*

*Proof.* Let $U, W \in V[G]$ be distinct normal measures on a cardinal $\kappa$. Let $U_0 = U \cap V$ and $W_0 = W \cap V$. Then $U_0, W_0 \in V$ generate $U, W$ respectively and therefore they are distinct normal measures in $V$. Suppose without loss of generality that $U_0 \lhd W_0$, and let $k : M^V_{W_0} \to j_W(V)$ be the factor map. The critical point of $k$ is greater than $\kappa$ (since both are $W_0$ and $W$ are normal). It follows that $U_0 \subseteq k(U_0) \in j_W(V) \subseteq M_W$. Since $V[G]$ and $M_W$ have the same subsets of $\kappa$, $k(U_0)$ also must generate $U \in M_W$. □

**Corollary 6.3.** *Assume UA and that $\kappa$ is the least measurable limit of measurables. Let $\vec{U}$ be the sequence of normal measures on all the measurables below $\kappa$. Let $G$ be $V$-generic for $\mathbb{P}_\kappa$. Then $V[G] \models$ weak UA $+ \neg$UA.*

*Proof.* We already proved that the Mitchell order is linear in $V[G]$, and therefore by Lemma 6.1, it remains to prove that every ultrapower by a $\sigma$-complete ultrafilter is equal to an ultrapower by a sum of normals. Let $U$ be a $\sigma$-complete ultrafilter. Since $\kappa$ is the least measurable in $V[G]$, $U$ is $\kappa$-complete. Then $U_0 = U \cap V$ is a $\kappa$-complete ultrafilter on $\kappa$ in $V$ so by Theorem 5.3 and Theorem 5.7 and Lemma 5.6 $V[G]_U = V[G]_W$ where $W$ is the sum of a finite iteration of normal ultrafilters.

To see that UA fails in $V[G]$, note that in $V[G]$, by Theorem 4.4, there is an ultrapower $M$ of the universe that admits a nontrivial elementary embedding $k : M \to M$. This would be impossible if UA held in $V[G]$. The reason is that assuming UA, by [6, Thm. 5.2] there is at most one elementary embedding $j : V[G] \to M$. Therefore $k \circ j = j$,. and so by a standard lemma on the Rudin–Keisler order (proved for example in [6, Cor. 4.29]), $k$ would be the identity. □

Our final application is to a natural question. Can two countably complete uniform ultrafilters on distinct cardinals have the same ultrapower? That is, given such ultrafilters $U_0, U_1$, can $V_{U_0}$ be equal to $V_{U_1}$?

**Proposition 6.4.** *Suppose that $U_0$ and $U_1$ are countably complete uniform ultrafilters on regular cardinals $\kappa_0$ and $\kappa_1$, and assume $V_{U_0} = V_{U_1}$. Then $\kappa_0 = \kappa_1$.*

*Proof.* Since $j_{U_0}$ and $j_{U_1}$ are elementary embeddings from $V$ into the same inner model, we can appeal to a theorem of Woodin [6, Theorem 3.4] to obtain that $j_{U_0} \restriction \mathrm{Ord} = j_{U_1} \restriction \mathrm{Ord}$. Assume without loss of generality that $\kappa_0 \leq \kappa_1$. Then since $U_1$ is uniform on $\kappa_1$, $j_{U_1}$ is discontinuous at $\kappa_1$. It follows that $j_{U_0}$ is discontinuous at $\kappa_1$. Since $\kappa_1$ is regular and $j_{U_0}$ is discontinuous at $\kappa_1$, we cannot have $\kappa_0 < \kappa_1$ by [5, Lemma 2.2.34]. Therefore $\kappa_0 = \kappa_1$. □

The following example shows that the assumption above that $\kappa_0$ and $\kappa_1$ are regular cardinals is necessary. It also demonstrates one of the complications arising in the attempt to extend our results on extensions of $\kappa$-complete ultrafilters on $\kappa$ to arbitrary countably complete ultrafilters.

For the remainder of the paper, let $\lambda$ be a measurable cardinal and let $\kappa$ denote the least limit of measurable cardinals of cofinality $\lambda$. Let $\Delta$ denote the set of measurable cardinals strictly between $\lambda$ and $\kappa$, and let $\vec{U} : \Delta \to V$ assign to each such measurable cardinal a normal ultrafilter. Finally, let $\mathbb{P}_\kappa$ be the Magidor iteration of Prikry forcings associated with $\vec{U}$ and let $G \subseteq \mathbb{P}_\kappa$ be $V$-generic.

**Lemma 6.5.** *If $D$ is a normal ultrafilter on $\lambda$, then $j_D^{V[G]} \restriction V = i \circ j_D$ where $i : V_D \to N$ is the complete iteration of $V_D$ by $j_D^V(\vec{U}) \restriction (\kappa, j_D^V(\kappa))$. Moreover $j_D^V(G) = (G \cap V_D) \times G_{\vec{s}}$ where $\vec{s}$ is the sequence of sets of indiscernibles associated with the complete iteration of $j_D^V(\vec{U}) \restriction (\kappa, j_D^V(\kappa))$.*

*Proof.* Note that $G \cap V_D$ is $V_D$-generic on $\mathbb{P}_\kappa(j_D(\vec{U}) \restriction \kappa)$, and therefore by the Mathias criterion and Lemma 3.4,

$$H = (G \cap V_D) \times G_{\vec{s}}$$

is $N$-generic on $i \circ j_D(\mathbb{P}_\kappa)$. Moreover, if $p \in G$, then $j_D^V(p) \restriction \kappa \in G \cap V_D$ and $j_D^V(p) \restriction (\kappa, j_D(\kappa))$ is a pure condition, and so $i(j_D^V(p)) \in (G \cap V_D) \times G_{\vec{s}}$. It follows that $i \circ j_D : V \to N$ lifts to an elementary embedding $j : V[G] \to N[H]$ with $j(G) = H$.

To show that $j = j_D^{V[G]}$ it suffices by Corollary 2.3 to show that $N[H] = \text{Hull}^{N[H]}(j[V[G]] \cup \{\lambda\})$. Since $i[V_D] = \text{Hull}^N(j[V] \cup \{\lambda\})$, we have

$$i[V_D] \cup \{G_{\vec{s}}\} \subseteq \text{Hull}^{N[H]}(j[V[G]] \cup \{\lambda\})$$

and $N[G_{\vec{s}}] \subseteq \text{Hull}^{N[H]}(j[V[G]] \cup \{\lambda\})$ by Lemma 3.4 applied in $V_D$. Since $N[H] = N[G_{\vec{s}}][G \cap V_D]$ and $N[G_{\vec{s}}] \cup \{G \cap V_D\} \subseteq \text{Hull}^{N[H]}(j[V[G]] \cup \{\lambda\})$, it follows that $N[H] = \text{Hull}^{N[H]}(j[V[G]] \cup \{\lambda\})$, as desired. $\square$

**Proposition 6.6.** *In $V[G]$, there are countably complete uniform ultrafilters on $\lambda$ and $\kappa$ with the same ultrapower.*

*Proof.* Let $D$ be a normal ultrafilter on $\lambda$. Let $f : \lambda \to V_\kappa$ be the increasing enumeration of $\vec{U}$, and let $U = [f]_D$. Then in $V_D$, $U$ is a normal ultrafilter on the least measurable $\gamma > \kappa$. Finally, let $W = \Sigma(D, U)$. Then $W$ is a uniform $\lambda$-complete ultrafilter on $\kappa^2$.

We claim that there is an extension $W^*$ of $W$ to a uniform $\lambda$-complete $V[G]$-ultrafilter on $\kappa^2$ such that $V[G]_{W^*} = V[G]_D$.

We will define $W^*$ as the analog of the canonical lift of $W$ (defined in Section 3.1) to this situation. The restricted ultrapower of $W^*$ will be the elementary embedding $i' \circ j_W^V$ where $i : V_W \to N$ is the complete iteration of $V_W$ by $j_W(\vec{U}) \restriction (\kappa, j_W(\kappa))$. The image generic will be $(G \cap V_W) \times G_{\vec{t}}$ where $\vec{t}$ is the sequence of sets of indiscernibles associated with the complete iteration of $j_W(\vec{U})$. The seed will just be $i([id]_W)$. These three ingredients uniquely determine $W^*$, and it is not hard to verify that a $V[G]$-ultrafilter $W^* \supseteq W$ with these invariants exists.

Note that the complete iteration of $V_W$ by $j_W(\vec{U}) \restriction (\kappa, j_W(\kappa))$ is just the tail of the complete iteration of $V_D$ by $j_D(\vec{U}) \restriction (\kappa, j_D(\kappa))$ after applying the first measure. Therefore by Lemma 6.5, $V[G]_{W^*} = N[(G \cap N) \times G_{\vec{t}}] = V[G]_D$, noting that the sequence $\vec{t}$ differs from the sequence $\vec{s}$ of sets of indiscernibles associated with $j_D(\vec{U}) \restriction (\kappa, j_D(\kappa))$ by just one ordinal.

It remains to show that $W^*$ is a uniform ultrafilter on $\kappa^2$. The reason is that $j_{W^*} \restriction V[G]_\kappa = j_D^{V[G]} \restriction V[G]_\kappa$, while $j_{W^*} \neq j_D^{V[G]}$, the latter following from the fact that

$$j_D^{V[G]}(G) = G \times G_{\vec{s}} \neq G \times G_{\vec{t}} = j_{W^*}(G)$$

If $W^*$ were not uniform on $\kappa^2$, then $W^*$ would be Rudin–Keisler equivalent to an ultrafilter $Z$ on some $\gamma < \kappa$ derived from $j_{W^*}$ and some $\xi < j_{W^*}(\gamma)$. But then $Z$ is also derived from $j_D$ and $\xi$, so $W^* \leq_{RK} Z \leq_{RK} D$. Since $D$ is normal and $W^*$ is nonprincipal, it follows that $D$ and $W^*$ are Rudin–Keisler equivalent, contrary to the fact that $j_{W^*} \neq j_D^{V[G]}$. □

## 7. Problems

We list out a few related problems we did not address:

**Question 7.1.** Can we characterize all the $\sigma$-complete extensions of a $\sigma$-complete ultrafilter on $V$ after the discrete Magidor iteration? In particular, are there only countably many extensions?

**Question 7.2.** Can we characterize the $\sigma$-complete extensions of sums of normals after other types of iterations of Prikry forcing?

**Question 7.3.** Working over any ground model $V$, can we find a characterization of all the extensions of a countably complete ultrafilter to a countably complete ultrafilter after the discrete Magidor iteration?

We conjecture that if $\kappa$ is strongly compact then after a discrete a Magidor iteration below $\kappa$, there is a $\kappa$-complete $V$-ultrafilter over $\kappa$ which has uncountably many lifts.

**Question 7.4.** Is there a forcing that preserves UA and adds a subset $X$ to the least supercompact cardinal $\kappa$ such that $X \notin V[Y]$ for any $Y \subseteq V$ of cardinality less than $\kappa$.

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