The Galvin property at successors of singulars

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- Some consistently new instances of $\lambda \to (\lambda, \omega + 1)$, relation to strong generating sequence of ultrafilters, and more...

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Question

Is it consistent to have a filter/ultrafilter U which is not of the previous form for which $Gal(U, \kappa, \kappa^+)$ holds?

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aThere are some limiting results due to Garti (see [6])

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Assuming larger cardinals, we were able to get this failure to hold globally, for every successor of singular cardinal.

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Let $\mathcal{C}$ be a witness for the strong negation. Then there exists $\mathcal{D}$, such that:

1. *$\mathcal{D}$ is also a witness for the strong negation;*
2. *For every normal ultrafilter U over κ, forcing with $Prikry(U)$ yields a generic extension where $\mathcal{D}$ cease to be a witness.*

Finish line

Thank you for your attention!

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