MATH 504 PROBLEM SET 7

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Let M be a countable transitive model of enough ZFC.

Problem 1. Let $\mathbb{P} \in M$ be a forcing notion. a condition $p \in \mathbb{P}$ is called an atom ant two extension $q, r \leq_{\mathbb{P}} p$ are compatible, namely $\neg(p \perp q)$. Show that if p is an atom then there is an M-generic filter $G \in M$ such that $p \in G$.

Problem 2. Show that if $\mathbb{P} \in M$ is atomless (namely splitting) then there are 2^ω-many distinct M-generic filters.

Problem 3. Prive that there is a bijection between (not necessarily generic) filters $G \subseteq Add(\omega, 1)$ and (partial) functions $f : \omega \to 2$.

Problem 4. Prove that if $S \subseteq \omega_1$ is a stationary set then S is fat, namely, for every $\alpha < \omega_1$ there is a closed $A \subseteq S$ such that $\text{otp}(A) = \alpha$ [Hint: by induction on α, the only problematic case is successor of a limit δ. In this case, set $C = \{\nu < \omega_1 \mid \exists A \subseteq S \text{ closed } \wedge ot(A) = \alpha \wedge \sup(A) = \nu\}$, prove that C is a club and take $\nu \in C \cap S$.]

Problem 5. Let $\mathbb{P}(S)$ be a forcing for shooting a club through the stationary set S. Let G be an M-generic filter. Prove that $C := \cup G$ is a club at ω_1 and $C \subseteq S$. [Hint: use the previous problem to show that C is unbounded.]

Problem 5. Prove that the following are equivalent for any filter $G \subseteq \mathbb{P} \in M$:

- G is M-generic.
- For any dense subset $D \subseteq \mathbb{P}$ such that $D \in M$ is open (namely if $p \in D$ and $q \leq_{\mathbb{P}p}$ then $q \in D$), $D \cap G \neq \emptyset$.
- G intersect every set $D \subseteq \mathbb{P}$, $D \in M$ which is pre-dense (namely if $\{q \in \mathbb{P} \mid \exists p \in D. q \leq_P p\}$ is a dense set)
- G intersect every $\mathcal{A} \subseteq \mathbb{P}$, $\mathcal{A} \in M$ which is a maximal antichain (namely, for every distinct $p, q \in \mathcal{A}$, $p \perp q$ and for every $r \in \mathbb{P}$ there is $p \in \mathcal{A}$ such that $\neg(r \perp p)$)

Problem 6. Let $\mathbb{P}$ be a forcing notion and let $D \subseteq \mathbb{P}$ be dense such that $D \in M$. We consider $D \cup \{1_{\mathbb{P}}\}$ as a forcing notion by restricting the order of $\leq_{\mathbb{P}}$ to D (we abuse notation and keep denoting it by $\leq_P$.

(a) Prove that if $G \subseteq \mathbb{P}$ is M-generic then $G \cap D$ is M-generic for $\langle D, \leq_{\mathbb{P}\rangle}$.

(b) Prove that if $H \subseteq D$ is M-generic then $G := \{q \in \mathbb{P} \mid \exists p \in H. q \geq_{\mathbb{P}p\}}$ is M-generic for $\mathbb{P}$.

Problem 7. We say that $\mathbb{P}_1, \mathbb{P}_2$ are isomorphic if there is $\pi : \mathbb{P}_1 \to \mathbb{P}_2$ which is order preserving, $1-1$ and $Im(\pi)$ is dense in $\mathbb{P}_2$. Prove that if

$\mathbb{P}_1 \simeq \mathbb{P}_2$ then for every M-generic filter G for $\mathbb{P}_1$, $\pi_*(G) := \{q \in \mathbb{P}_2 \mid \exists p \in \mathbb{P}_1 . q \geq_{\mathbb{P}_2 \pi(p)\}}$ is M-generic for $\mathbb{P}_2$.

Problem 8. Recall that $Add(\omega, \omega_2) := \{f : \omega_2 \times \omega \to 2 \mid |f| < \omega\}$. Let G be M-generic for $Add(\omega, \omega_2)$.

(a) Prove that for every $\alpha < \omega_2$, $\cup G$ is a function $\cup G : \omega_2 \times \omega \to 2$.

(b) Define $g_\alpha : \omega \to 2$ by $g_\alpha(b) = (\cup G)(\alpha, n)$. Prove that if $\alpha \neq \beta$ then $g_\alpha \neq g_\beta$.

Remark 0.1. This is the forcing we are going to use to prove the consistency of $ZFC + \neg CH$. You can imagine that if $M[G]$ is the ZFC model which if produces from M and G then in $M[G]$ we will have ω_2-many different function from ω to 2 and therefore $2^\omega \geq \omega_2$. However, there is another delicate point, as cardinals are not absolute notion even for transitive models. So all we get so far is that in $M[G]$, $2^\omega \geq (\omega_2)^M$. The next crucial step would be to prove that $(\omega_2)^M = \omega_2$ and then $M[G] \models \neg CH$.