MATH 504 PROBLEM SET 6

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Problem 1. Prove that ZF is not finitely axiomatizable.

Problem 2. Prove by induction that $L_\alpha \cap On = \alpha$. [Hint: At successor steps, show that α is definable from L_α.]

Problem 3. On this exercise we shall define the set $D(A)$ formally.

(a)

Definition 0.1. Let:
(i) $Diag_\in(A, n, i, j) = \{s \in A^n \mid s(i) \in s(j)$.
(ii) $Diag_=(A, n, i, j) = \{s \in A^n \mid s(i) = s(j)\}$.
(iii) $Proj(A, R, n) = \{s \in A^n \mid \exists r \in R.r \restriction n = s\}$

Prove that these defined notions are absolute for transitive models.

(b)

Definition 0.2. Define $D'(k, A, n)$ recursively on k (for all n):
(i) $D'(0, A, n) = \{Diag_\in(A, n, i, j), Diag_=(A, n, i, j) \mid i, j < n\}$.
(ii) $D'(k+1, A, n) = D'(k, A, n) \cup \{A^n \setminus R \mid R \in D'(k, A, n)\} \cup \{R \cap S \mid R, S \in D'(k, A, n)\} \cup \{Proj(A, R, n) \mid R \in D'(k, A, n+1)\}$

Again, prove that $D'(k, A, n)$ is absolute for transitive models.

(c)

Definition 0.3. $Df(A, n) := \cup_{k<\omega} D'(k, A, n)$.

Prove that $Df(A, n)$ is absolute for transitive models.

[Remark: The idea of $Df(A, n)$ is that if R is an n-arry relation defined from A without parameters, then $R \in Df(A, n)$. By the usual way formulas are defined, the set $Df(A, n)$ is exactly those R's. We have that $|Df(A, n)| \leq \omega$.]

(d)

Definition 0.4. Let $\mathcal{D}(A) := \{X \in P(A) \mid \exists n < \omega.\exists R \in D(A, n+1)\exists s \in A^n.X = \{x \in A \mid s^\frown x \in R\}\}$.

Prove that $\mathcal{D}(A)$ is absolute for transitive models.

Problem 4. Prove that for every $\alpha > \omega$, $|L_\alpha| = |V_\alpha|$ iff $\alpha = \beth_\alpha$.

Problem 5. Assume that $V = L$, prove that for every $\alpha > \omega$, $L_\alpha = V_\alpha$ iff $\alpha = \beta_\alpha$.

Problem 6. Let A be any set. Define the model $L(A)$ as follows:

$L_0(A) = \{A\} \cup tr(A)$

$L_{\alpha+1}(A) = D(L_\alpha(A))$

δ is limit $\Rightarrow L_\delta(A) = \cup_{\alpha<\delta} L_\alpha(A)$
$L(A) = \cup_{\alpha\in On} L_\alpha(A)$.

(a) Prove that $L(A)$ is a transitive model of ZF.
(b) Prove that $L(A) \models AC$ iff $tr(A)$ has a well-ordering in $L(A)$.
(c) Prove that $L(A)$ is the least transitive model M of ZF such that $A \in M$.

[Remark: $L(\mathbb{R})$ need not satisfy AC]

Problem 7. Consider the language $\mathcal{L} = \{E, P\}$ where E is a binary relation and P is a unary predicate. For a model M and a set $U \subseteq M$, we consider the model $(M, \in, U)$ for the language $\mathcal{L}$. We say that B is definable in a model $(M, \in, U)$ if there is an $\mathcal{L}$-formula $\phi(x, x_1, ..., x_n)$ and $a_1, ..., a_n \in M$ such that $B = \{x \in M \mid (M, \in, U) \models \phi(x, a_1, ..., a_n)\}$. For any two sets A, U, we let

$$D_U(A) = \{B \in P(A) \mid B \text{ is definable in } (A, \in, U \cap A)\}$$

Let U be any set. Define $L[U]$ as follows:
$L_0[U] = \emptyset$
$L_{\alpha+1}[U] = D_U(L_\alpha[U])$
δ is limit $\Rightarrow L_\delta[U] = \cup_{\alpha<\delta} L_\alpha[U]$
$L[U] = \cup_{\alpha\in On} L_\alpha[U]$.

(1) Prove that if M is transitive then $M \cup \{M\} \cup \{U \cap M\} \subseteq D_U(M)$.
(2) Prove that $L[U] = L[U \cap (L[U])]$ [Hint: Let $\bar{U} = U \cap L[U]$, by induction on α prove that $L_\alpha[U] = L_\alpha[\bar{U}]$.]
(3) Prove that $L[U] \models ZFC$.
(4) Prove that $L[U]$ is the minimal transitive ZFC model M satisfying $On \subseteq M$ and $U \cap L[U] \in M$.