MATH 504 PROBLEM SET 4

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Problem 1. Prove that $cf(2^{<\kappa}) \geq cf(\kappa)$ and prove that under GCH, $cf(2^{<\kappa}) = cf(\kappa)$.

Problem 2. Prove Hausdorff's formula:

$$\forall \alpha, \beta,\ \aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}$$

Problem 3. Define β_α by induction on α:

$$\beth_0 = \aleph_0,\ \beth_{\alpha+1} = 2^{\beth_\alpha}$$

and for limit δ,

$$\beth_\delta = \sup_{\alpha<\delta} \beth_\delta$$

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(a) Prove that GCH is equivalent to the statement $\forall \alpha . \aleph_\alpha = \beth_\alpha$.

(b) Prove that α is weakly inaccessible iff α is regular and

$$\forall \beta < \alpha . \aleph_\beta < \alpha$$

(c) Prove that α is strongly inaccessible iff α is regular and

$$\forall \beta < \alpha . \beth_\beta < \alpha$$

(d) Conclude that under GCH weakly and strongly inaccessible coincide.

Problem 4. Suppose that $\langle \kappa_i \mid i < \lambda \rangle$ is a sequence of cardinals. Define:

$$\sum_{i<\lambda} \kappa_i = |\biguplus_{i<\lambda} \{i\} \times \kappa_i|$$

$$\prod_{i<\lambda} \kappa_i = \{f \in {}^\lambda(\sup_{i<\lambda} \kappa_i) \mid \forall i < \lambda . f(i) < \kappa_i\}$$

Denote by $\kappa = \sup_{i<\lambda} \kappa_i$.

(a) Prove that $\sum_{i<\lambda} \kappa_i = \lambda \cdot \kappa$.[Hint: Prove a double inequality.]

(b) $\prod_{i<\lambda} \kappa_i = (\kappa)^\lambda$ [Hint: One direction is easy. For the other direction, split $\lambda = \uplus_{i<\lambda} A_i$ where $|A_i| = \lambda$ (just since $|\lambda \times \lambda| = |\lambda|$) and prove: $\prod_{i<\lambda} \kappa_i = \prod_{i<\lambda}(\prod_{j\in A_i} \kappa_j) \geq \prod_{i<\lambda} \kappa = \kappa^\lambda$]

(c) Prove that $\tau^{\sum_{i<\lambda} \kappa_i} = \prod_{i<\lambda} \tau^{\kappa_i}$.

(d) Prove the following variation of König's Theorem: If for every $i \in I$, $\kappa_i < \lambda_i$ then

$$\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i$$

[Hint: Suppose that there is an onto function $F : \biguplus_{i\in I}\{i\}\times\kappa_i \to \prod_{i\in I}\lambda_i$, and let $Z_i = F[\{i\}\times\kappa_i]$, now use $|Z_i| \leq \kappa_i < \lambda_i$ to "diagonalize", i.e. find a function $g \in \prod_{i\in I} Z_i$ such that for every $i \in I$ and every $f \in Z_i$, $g(i) \neq f(i)$. Using g derive the usual contradiction.]

(e) Conclude that for every cardinal κ, $2^\kappa > \kappa$. [Hint: $1+1+1+1...+1 < 2\cdot 2\cdot 2....\cdot 2$.]

(f) Conclude our version of König's lemma that if $\lambda \geq cf(\kappa)$ then $\kappa^\lambda > \kappa$.

Problem 5. Prove that the following sets are clubs at κ:

(a) $\{\omega^\alpha \mid \alpha < \kappa\}$ (here ω^α is computed using ordinal exponentiation).

(b) $cl(A)$ for an unbounded set A.

(c) For a limit cardinal κ (of uncountable cofinality), $\{\alpha < \kappa \mid \alpha \text{ is a cardinal}\}$.

(d) If κ is strongly inaccessible then $\{\alpha < \kappa \mid \alpha \text{ is a strong limit cardinal}\}$ is a club.

(e) Prove or disprove: If A is not a club and B is not a club then $A \cap B$ is not a club.

(f) Prove or disprove: If A is disjoint to a club and B is disjoint to a club then $A \cup B$ is disjoint to a club

Problem 6. Let κ be a regular cardinal and $f : \kappa^n \to \kappa$ be any n-ary function. Denote by

$$C_f = \{\alpha < \kappa \mid f''\alpha^n \subseteq \alpha\}$$

where $\kappa^n = \underbrace{\kappa\times\kappa\times ...\times\kappa}_{n\text{-times}}$.

(a) Prove that C_f is a club at κ.

(b) Prove that if C is a club at κ then there is $f : \kappa \to \kappa$ such that $C_f \subseteq C$ [Hint: Let $f : \kappa \to C$ be an order isomorphism (prove first that $otp(C, <) = \kappa$) then prove that $C_f \subseteq C$.]