MATH 504 PROBLEM SET 3

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Problem 1. Prove that there is a n ordinal α such that $\omega^\alpha = \omega$ (here ω^α stands for ordinal exponentiation) and prove that the minimal such ordinal is countable.

Problem 2. Prove Using Zorn's lemma that for every infinite set A, $A \times A \sim A$. [Hint: consider $\Sigma = \{B \subseteq A \mid B \times B \sim B\}$ ordered by inclusion. Prove that this order satisfies the assumptions of Zorn's lemma and let B^* be maximal. Then $B^* \times B^* \sim B^*$. Split into cases: $A \setminus B^* \precsim B^*$, in this case prove that $A \sim B^*$ and conclude that $A \times A \sim A$. In the second case, choose $X \subseteq A \setminus B^*$ of cardinality B^* and prove that $(B^* \cup X) \times (B^* \cup X) \sim B^* \cup X$.]

Problem 3. For an ordinals α, consider the following order on $\alpha \times \alpha$:

$$\langle x, y\rangle \prec \langle a, b\rangle \Leftrightarrow (\max(x,y) < \max(a,b)) \vee (\max(x,y) = \max(a,b) \wedge (x,y) <_{Lex} (a,b)$$

you already prove that this is a well order.

(a) Prove by transfinite induction on α that $otp(\aleph_\alpha \times \aleph_\alpha, \prec) = \aleph_\alpha$. [For limit cardinals use $\aleph_\delta \times \aleph_\delta = \cup_{\alpha<\delta} \aleph_\alpha \times \aleph_\alpha$ and at successor step $\aleph_{\alpha+1}$, not that $\gamma := otp(\aleph_{\alpha+1} \times \aleph_{\alpha+1}, \prec) \geq \aleph_\alpha$ and assume toward a contradiction that $\gamma > \aleph_{\alpha+1}$. there is a pair $\langle \beta, \rho\rangle \in \aleph_{\alpha+1} \times \aleph_{\alpha+1}$ such that $otp(\beta \times \gamma, \prec) = \aleph_{\alpha+1}$, then pick $\delta = \max(\beta, \gamma)$ and prove using the induction hypothesis that $\aleph_{\alpha+1} \leq |\delta \times \delta| = \aleph_\alpha$, and derive a contradiction.]

(b) Conclude using the axiom of choice again that for every set S, $A \times A \sim A$.

Problem 4. Prove that $\mathbb{R} \times \mathbb{R} \sim \mathbb{R}$ [Hint: Use the fact that $\mathbb{R} \sim {}^{\mathbb{N}}\{0,1\}$ and construct a function witnessing ${}^{\mathbb{N}}\{0,1\} \times {}^{\mathbb{N}}\{0,1\} \sim {}^{\mathbb{N}}\{0,1\}$]

Problem 5. Let $r_1, r_2 \in \mathbb{R}$ (be Dedekind cuts) define

$$r_1 + r_2 = \{x + y \mid x \in r_1, y \in r_2\}$$

Prove that

(a) $r_1 + r_2$ is a Dedekind cut.

(b) $r + 0 = r$.

(c) $(r_1 + r_2) + r_3 = r_1 + (r_2 + r_3)$.

Problem 6. Prove that if $\langle A, R\rangle$ is a separable order (namely a total order with a countable dense subset) then $|A| \leq 2^{\aleph_0}$.

Problem 7. Prove that for every $a, b \in \mathbb{Q}$, if $a < b$ then $\mathbb{Q} \cap (a,b)$ is order isomorphic to $\mathbb{Q}$. [Hint: Cantor's theorem.]

Problem 8. Prove that for every $\alpha < \omega_1$ there is $X_\alpha \subseteq \mathbb{Q}$ such that $otp(X_\alpha, <) = \alpha$. [Hint: Prove it by transfinite induction, and the previous exercise.]