MATH 504 PROBLEM SET 2

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(1) Prove the lemma from class: Let $\langle A, R\rangle$ be a well-ordering and $X \subseteq A$. Then X is an initial segment iff $X = A$ or $\exists x \in A. A_R[x] = X$. Hint: define $x = \min A \setminus X$

(2) Show that $\alpha < \beta$ implies that $\gamma + \alpha < \gamma + \beta$ and $\alpha + \gamma \leq \beta + \gamma$. Give an example to show that the "$\leq$" cannot be replaced by "$<$". Also, show:

$$\alpha \leq \beta \rightarrow \exists! \delta \, (\alpha + \delta = \beta).$$

(3) Show that if $\gamma > 0$, then $\alpha < \beta$ implies that $\gamma \cdot \alpha < \gamma \cdot \beta$ and $\alpha \cdot \gamma \leq \beta \cdot \gamma$. Give an example to show that the "$\leq$" cannot be replaced by "$<$". Also, show:

$$(\alpha \leq \beta \wedge \alpha > 0) \rightarrow \exists! \delta, \xi \, (\xi < \alpha \wedge \alpha \cdot \delta + \xi = \beta).$$

(4) Verify that ordinal exponentiation satisfies:

$$\alpha^{\beta+\gamma} = \alpha^{\beta} \cdot \alpha^{\gamma} \quad \text{and} \quad (\alpha^{\beta})^{\gamma} = \alpha^{\beta \cdot \gamma}.$$

(5) Let α be a limit ordinal. Show that the following are equivalent:
(a) $\forall \beta, \gamma < \alpha (\beta + \gamma < \alpha)$.
(b) $\forall \beta < \alpha \, (\beta + \alpha = \alpha)$.
(c) $\forall X \subset \alpha \big(\text{type}(X) = \alpha \vee \text{type}(\alpha \setminus X) = \alpha\big)$.
(d) $\exists \delta \, (\alpha = \omega^{\delta})$ (ordinal exponentiation).
Such α are called *indecomposable*.

(6) Prove the Cantor Normal Form Theorem for ordinals: Every non-0 ordinal α may be represented in the form:

$$\alpha = \omega^{\beta_1} \cdot l_1 + \cdots + \omega^{\beta_n} \cdot l_n,$$

where $1 \leq n < \omega$, $\alpha \geq \beta_1 > \cdots > \beta_n$, and $1 \leq l_i < \omega$ for $i = 1, \ldots, n$. Furthermore, this representation is unique. α is called an *epsilon number* iff $n = 1$, $l_1 = 1$, and $\beta_1 = \alpha$ (i.e., $\omega^{\alpha} = \alpha$). Show that if κ is an uncountable cardinal, then κ is an epsilon number and there are κ epsilon numbers below κ; in particular, the first epsilon number, called ϵ_0, is countable. All exponentiation is ordinal exponentiation in this exercise.

(7) a. $\alpha + \beta = \text{type}(\alpha \times \{0\} \cup \beta \times \{1\}, R)$, where

$$R = \{\langle\langle\xi, 0\rangle, \langle\eta, 0\rangle\rangle : \xi < \eta < \alpha\} \cup$$
$$\{\langle\langle\xi, 1\rangle, \langle\eta, 1\rangle\rangle : \xi < \eta < \beta\} \cup [(\alpha \times \{0\}) \times (\beta \times \{1\})].$$

(7) b. $\alpha \cdot \beta = \text{type}(\beta \times \alpha, R)$, where R is lexicographic order on $\beta \times \alpha$:

$$\langle\xi, \eta\rangle R \langle\xi', \eta'\rangle \leftrightarrow (\xi < \xi' \vee (\xi = \xi' \wedge \eta < \eta')).$$

(7) Prove that the following definition of ordinal exponentiation is equivalent to Definition 9.5: Let

$$F(\alpha, \beta) = \{f \in {}^{\beta}\alpha : |\{\xi : f(\xi) \neq 0\}| < \omega\}.$$

If $f, g \in F(\alpha, \beta)$ and $f \neq g$, say $f \lhd g$ iff $f(\xi) < g(\xi)$, where ξ is the largest ordinal such that $f(\xi) \neq g(\xi)$. Then $\alpha^\beta = \text{type}(\langle F(\alpha, \beta), \lhd\rangle)$.

(8) Prove that if α is an ordinal and $x \in \alpha$ then x is an ordinal and $x = \alpha_{\in}[x]$.

(9) Prove the following:

1. If α is an ordinal then $\emptyset \leq \alpha$.
2. If α is an ordinal then $\alpha + 1 := \alpha \cup \{\alpha\}$ is an ordinal and is the successor of α in the sense that it is the minimal ordinal greater than α.
3. If A is a set of ordinals without a greatest element then $\sup A := \cup A$ is an ordinal strictly greater then all the ordinals in A.