RESEARCH STATEMENT

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My research interests are **representation theory**, **algebraic combinatorics**, **Lie (super) algebras** and their connections. I look forward to exploring broader impacts of representation theory, e.g. in geometry and number theory in the future.

Broadly speaking, representation theory studies abstract algebraic structures by representing their elements as concrete linear maps. An important branch is *Lie theory* which studies Lie groups, Lie algebras, and generalizations such as Lie superalgebras and Kac–Moody algebras. Algebraic combinatorics often employs representation theory to investigate objects of combinatorial nature, e.g., symmetric polynomials. For example, Schur polynomials emerge, surprisingly, as characters of symmetric groups, unitary groups, general linear groups and Lie algebras ("Type A" objects); other symmetric polynomials are often parametrized by partitions, which are ubiquitous in representation theory. The symbiotic interplay of representation theory and algebraic combinatorics is often reciprocal: advances in one field eventually shed light on the other.

In my research, I often rely on (1) observations as a result of explicit computations, and (2) representation theoretic tools such as the highest weight theory. My *thesis project* [Zhu22, SZ23] connects a family of symmetric polynomials with the super Shimura operators. These operators were first studied by Shimura in [Shi90]. My work gives *super* analogs of results by Sahi and Zhang in [SZ19]. In a different direction, I study *root strings in the direction of imaginary roots in Kac–Moody algebras* [CCM$^+$24]. We show the existence of infinite root strings in symmetrizable Kac–Moody algebras, and find specific bounds for the growths of the root multiplicities along these root strings. In this statement, I will review my work on the respective projects (Sections 1 and 2) and introduce the main results (Subsections 1.2 and 2.2). I will also talk about future directions.

1. Super Shimura Operators and Interpolation Polynomials

1.1 Introduction In [Shi90], Shimura introduced a basis for the algebra of invariant differential operators on a Hermitian symmetric space, and formulated the problem of determining their eigenvalues, or equivalently their images under the Harish-Chandra homomorphism. These operators are now known as Shimura operators, and can be realized using the pair of Lie algebras $(\mathfrak{g}, \mathfrak{k})$ associated with the symmetric space. In [SZ19], Sahi and Zhang proved that these images are given by Okounkov interpolation polynomials of Type BC [Oko98] for appropriate choices of parameters. These were generalized by Sergeev and Veselov's work [SV09] in the setting of Type BC supersymmetry. Then it is natural to ask what if $(\mathfrak{g}, \mathfrak{k})$ is a Hermitian superpair of Lie superalgebras. The following problems are considered:

- **P1.** Determine the eigenvalues of super Shimura operators using Sergeev and Veselov polynomials.
- **P2.** Characterize the spherical representations for Hermitian symmetric superpairs.

P1 is the ultimate goal. It degenerates to the question originally posted by Shimura in the non-super setting. In [Zhu22], **P2** is used as one of the main tools to answer **P1**. A complete answer to **P2** would naturally extend the classical Cartan–Helgason Theorem on spherical representations. But **P2** remains an open conjecture. In [SZ23], a work-around is proposed to fully answer **P1** in Type A. The vertical arrows in the following diagram represent **P1** in both super and non-super scenarios.

$$\begin{array}{ccc}
\text{Okounkov Polynomials} & \xrightarrow{\text{Superization}} & \text{Sergeev–Veselov Polynomials} \\
\Big\updownarrow {\scriptstyle [\text{SZ19}]} & & \Big\updownarrow {\scriptstyle [\text{Zhu22, SZ23}]} \\
\text{Shimura Operators} & \xrightarrow{\text{Superization}} & \text{Super Shimura Operators}
\end{array}$$

Let $(\mathfrak{g}, \mathfrak{k})$ be a Hermitian symmetric superpair with the Harish-Chandra decomposition $\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$ [Zhu22, SZ23]. The *super Shimura operators* $\mathscr{D}_\mu$ are defined in $\mathfrak{D} = \mathfrak{U}^{\mathfrak{k}}/(\mathfrak{U}\mathfrak{k})^{\mathfrak{k}}$, where $\mathfrak{U}^{\mathfrak{k}}$ the $\mathfrak{k}$-centralizer in the universal enveloping algebra $\mathfrak{U} = \mathfrak{U}(\mathfrak{g})$. These $\mathscr{D}_\mu$ are the super Shimura operators, whose definition involves the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$ and the multiplicity-free $\mathfrak{k}$-decompositions of $\mathfrak{S}(\mathfrak{p}^+), \mathfrak{S}(\mathfrak{p}^-)$, naturally indexed by $\mu \in \mathscr{H}(p, q)$ [CW01].

Using Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, we define the *Harish-Chandra isomorphism associated with* $(\mathfrak{g}, \mathfrak{k})$ on $\mathfrak{D}$, denoted as γ^0. It captures the supersymmetry of the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$, and maps to the polynomial algebra on $\mathfrak{a}^*$, $\mathfrak{P}(\mathfrak{a}^*)$.

Another ingredient in our recipe is the *Type BC supersymmetric interpolation polynomials* first introduced by Sergeev and Veselov in [SV09]. Essentially, they are special even supersymmetric polynomials with prescribed zeros. Let $\Lambda^0(\mathfrak{a}^*)$ be the ring of even supersymmetric polynomials on $\mathfrak{a}^*$. In $\Lambda^0(\mathfrak{a}^*)$ for each $\mu \in \mathscr{H}(p, q)$, there exists a unique degree $2|\mu|$ polynomial P_μ, properly normalized, satisfying the *vanishing conditions*

$$P_\mu(2\overline{\lambda^\natural} + \rho) = 0, \text{ for any } \lambda \in \mathscr{H}(p, q) \text{ such that } |\lambda| \leq |\mu|, \lambda \neq \mu. \tag{1}$$

Here ρ is the Weyl vector of $\Sigma(\mathfrak{g}, \mathfrak{a})$, and the notation $2\overline{\lambda^\natural}$ refers to some specific coordinates associated with λ. The choice of coordinates is not essential.

1.2 Results My papers [Zhu22, SZ23] solve **P1** for $\mathfrak{g} = \mathfrak{gl}(2p|2q)$ and $\mathfrak{k} = \mathfrak{gl}(p|q) \oplus \mathfrak{gl}(p|q)$. [Zhu22] also gives a partial answer to **P2**.

Theorem 1.1 ([SZ23])**.** *The Harish-Chandra image $\gamma^0(\mathscr{D}_\mu)$ is equal to some non-zero multiple of P_μ.*

We check two assertions: (A) $\operatorname{Im} \gamma^0 \subseteq \Lambda^0(\mathfrak{a}^*)$, and (B) $\gamma^0(\mathscr{D}_\mu)$ satisfies the vanishing condition (1). (B) is considerably harder than (A). We discuss how we prove both claims.

On the center $\mathfrak{Z}$ of $\mathfrak{U}(\mathfrak{g})$, we have the usual *Harish-Chandra isomorphism* γ and it captures the supersymmetry of $\Sigma(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}_+$ is a Cartan subalgebra of $\mathfrak{g}$. The projection $\mathfrak{h} \to \mathfrak{a}$ induces a restriction map on the polynomial algebra on $\mathfrak{h}^*$. We denote its restriction to $\Lambda(\mathfrak{h}^*)$ (the ring of supersymmetric polynomials on $\mathfrak{h}^*$) as $\mathtt{Res}$. Let $\pi : \mathfrak{Z} \hookrightarrow \mathfrak{U}^{\mathfrak{k}} \to \mathfrak{D}$ be the quotient map. We first show that the following diagram commutes.

$$\begin{array}{ccc}
\mathfrak{Z} & \xrightarrow{\ \pi\ } & \mathfrak{D} \\
{\scriptstyle \gamma}\Big\downarrow & & \Big\downarrow{\scriptstyle \gamma^0} \\
\Lambda(\mathfrak{h}^*) & \xrightarrow{\ \mathtt{Res}\ } & \Lambda^0(\mathfrak{a}^*)
\end{array} \tag{2}$$

In particular, we show γ^0 is an isomorphism and $\operatorname{Im} \gamma^0 = \Lambda^0(\mathfrak{a}^*)$, from which Assertion (A) follows. Then we prove $\mathtt{Res}$ is surjective, from which we obtain the following theorem.

Theorem 1.2 ([SZ23])**.** *The quotient map π is surjective onto $\mathfrak{D} = \mathfrak{U}^{\mathfrak{k}}/(\mathfrak{U}\mathfrak{k})^{\mathfrak{k}}$.*

By Theorem 1.2, there exists $Z_\mu \in \mathfrak{Z}$ so that $\pi(Z_\mu) = \mathscr{D}_\mu$. By (2),

$$\gamma^0(\mathscr{D}_\mu) = \mathtt{Res}(\gamma(Z_\mu)). \tag{3}$$

Thus to pin down (B), we study the action of Z_μ on some $\mathfrak{k}$-spherical $\mathfrak{g}$-modules. To this end, we construct the *generalized Verma modules* $I(\lambda) := \operatorname{Ind}_{\mathfrak{k}+\mathfrak{p}^+}^{\mathfrak{g}} W(\lambda^\natural)$, induced from $W(\lambda^\natural)$, an irreducible $\mathfrak{k}$-module appearing in the aforementioned multiplicity-free decomposition of $\mathfrak{S}(\mathfrak{p}^+)$. Here the action of $\mathfrak{k}$ on $W(\lambda^\natural)$ is trivially extended to $\mathfrak{p}^+$.

Theorem 1.3 ([SZ23])**.** *The central element Z_μ acts on $I(\lambda)$ by 0 whenever $|\lambda| \leq |\mu|$ and $\lambda \neq \mu$.*

Therefore, Z_μ acts by 0 on $I(\lambda)$ by Theorem 1.3. Moreover, the irreducible $\mathfrak{g}$-module of the same highest weight is a quotient of $I(\lambda)$, and thus Z_μ inherits such vanishing conditions. By (3), we prove Assertion (B) and hence Theorem 1.1.

The following result is used to prove Theorem 1.2, and was obtained in an earlier paper [Zhu22]. Let Γ be the Harish-Chandra homomorphism defined on $\mathfrak{U}^{\mathfrak{k}}$ instead of $\mathfrak{U}^{\mathfrak{k}}/(\mathfrak{U}\mathfrak{k})^{\mathfrak{k}} = \mathfrak{D}$. Let ρ be the Weyl vector of the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$.

Theorem 1.4 ([Zhu22])**.** *Suppose a $\mathfrak{g}$-module V is of highest weight $\lambda \in \mathfrak{h}^*$ and $\dim V^{\mathfrak{k}} = 1$. Let $D \in \mathfrak{U}^{\mathfrak{k}}$. Then D acts on $V^{\mathfrak{k}}$ by the scalar $\Gamma(D)\,(\lambda|_{\mathfrak{a}} + \rho)$.*

In [Zhu22], a different method is considered to prove (B). Specifically, (B) follows from Theorem 1.4 by assuming the sphericity of certain irreducible finite dimensional $\mathfrak{g}$-modules. In fact, this "superizes" the original method used in [SZ19] to answer the non-super **P1**. Such method naturally leads to **P2**. Let $V(\lambda^\natural)$ be the irreducible $\mathfrak{g}$-module of the same highest weight as $W(\lambda^\natural)$ (with respect to a compatible Borel subalgebra).

Conjecture 1 (**P2**)**.** *Every irreducible $\mathfrak{g}$-module $V(\lambda^\natural)$ is spherical for $\lambda \in \mathscr{H}(p,q)$.*

When q is 0, this degenerates to the non-super scenario, and is fully answered by the Cartan–Helgason Theorem. It characterizes $\mathfrak{k}$-spherical irreducible $\mathfrak{g}$-modules by giving a necessary and sufficient condition on its highest weight restricted to $\mathfrak{a}$. In [Zhu22], I showed that Conjecture 1 is true when $p = q = 1$:

Theorem 1.5 ([Zhu22])**.** *For $p = q = 1$, all the irreducible $\mathfrak{g}$-modules $V\left(\lambda^\natural\right)$ are spherical.*

I introduced the concept of *quasi-spherical* vectors of a module. A non-zero vector $v \in V$ is called quasi-spherical if (1) $\mathfrak{k}.v$ is contained in the maximal submodule, and (2) v generates V. Thus v descends to a spherical vector in the irreducible quotient. Specifically, the *Kac modules* $K(\check{\lambda})$ were considered. These $K(\check{\lambda})$ have $V\left(\lambda^\natural\right)$ as their irreducible quotients. I showed $K(\check{\lambda})$ are indeed quasi-spherical. The proof is computational.

Theorem 1.6 ([Zhu22])**. Theorem 1.1** *follows from* **Conjecture 1**.

1.3 Future Directions I expect the theory to work for superpairs constructed using Jordan superalgebras as in, e.g. [SSS20]. In this general setting, I would like to

(1) define super Shimura operators and study **P1**.
(2) prove Theorem 1.2 and 1.3.
(3) give a full answer to **P2**, which will
  (a) completely generalize [AS15] in this case; and
  (b) allow us to answer **P1** differently.
(4) find the specific scalar in Theorem 1.1 in terms of supersymmetric polynomials as in [SZ19].

Let me point out that this project bears resemblance to other theories too, e.g. the super version of the Capelli eigenvalue problem presented in [SSS20] which generalizes [KS93, Sah94] in the classical picture.

2. Root Strings in Kac–Moody Algebras

2.1 Introduction Let $\mathfrak{g} = \mathfrak{g}(A)$ be a Kac–Moody algebra where A is a generalized Cartan matrix. It is well-known that any Kac–Moody algebra can be described by its root space decomposition [Kac90]. Let Δ denote its root system and set $\bar{\Delta} := \Delta \cup \{0\}$. When A is positive definite, $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra. In such cases, the length of a root string is at most 4; in fact, this happens in the rank 2 example of G_2 [Hum78]. Also, the multiplicity of a root is always 1, which also makes the $\mathfrak{sl}(2)$-module structure of any root string straightforward. A natural question to ask is what these root strings look like in general.

Dropping the condition that A is positive definite, $\mathfrak{g}$ becomes infinite dimensional. Let Δ^{re} and Δ^{im} be the sets of real and imaginary roots respectively. Given two roots $\alpha, \beta \in \Delta$, we consider the *β-string through α*, i.e. $R_\alpha(\beta) := \{\alpha + i\beta : i \in \mathbb{Z}\} \cap \bar{\Delta}$. If $R_\alpha(\beta)$ contains at least 2 roots, then it is called *non-trivial*. If $R_\alpha(\beta)$ is infinite in at least one direction, it is called *infinite*. If $R_\alpha(\beta)$ is infinite in exactly one direction, we call it *semi-infinite*. If it extends infinitely in both directions, we call it *bi-infinite*. By the *growth* of $R_\alpha(\beta)$, we mean the behavior of the function $\dim \mathfrak{g}_{\alpha+i\beta}$ in i.

In parallel with the semisimple cases in [Hum78], any root string along the direction of a real root is finite and the root multiplicity is always 1 for real roots [Kac90]. Morita showed [Mor88] that $A_{ij} = -1$ and $A_{ji} < -1$ for some i, j if and only if $|\{\alpha + k\beta : k \in \mathbb{Z}\} \cap \Delta^{\text{re}}| \in \{3, 4\}$ for some $(\alpha, \beta) \in \Delta \times \Delta^{\text{re}}$. However, the questions about lengths, multiplicities, and structures of root strings remained unanswered in general.

Now let us suppose that β is imaginary. We study the following questions:

Q1. How "long" can a β-string $R_\alpha(\beta)$ be?

Q2. What can we say about the growth along this string?

2.2 Results The symmetrizability of A guarantees the existence of a non-degenerate symmetric form $(\cdot, \cdot)$ on $\mathfrak{h}^*$. Then $\beta \in \Delta_{\text{im}}$ if and only if $(\beta, \beta) \le 0$. We say a root β is *isotropic* if $(\beta, \beta) = 0$. Otherwise it is *non-isotropic*. Depending on the conditions, there are three major cases, and we employed different techniques to investigate each of these unique situations. Our results are summarized in the following table.

Condition		**Cardinality (Q1)**	**Growth (Q2)**
β is non-isotropic		Infinite	At least exponential
β is isotropic	$(\alpha, \beta) = 0$	Bi-infinite	Bounded
	$(\alpha, \beta) \neq 0$	Semi-infinite	At least subexponential

Various cases of $R_\alpha(\beta)$

We first study the case when β is non-isotropic. The following result is a direct application of [Mar21, Corollary C].

Proposition 2.1 ([CCM+24]). *Let $\beta \in \Delta_{\text{im}}$ be non-isotropic, and let $\alpha \in \bar{\Delta}$. If $R_\alpha(\beta)$ is non-trivial, then at least one of $\alpha \pm \mathbb{N}\beta$ is contained in $R_\alpha(\beta)$.*

Then by a result of Kac [Kac90, Corollary 9.12], we show that there exists a free Lie subalgebra in $\bigoplus_{k\in\mathbb{N}} \mathfrak{g}_{k\beta}$. Free Lie algebras always possess exponential growth, and this fact leads to the following theorem.

Theorem 2.2 ([CCM+24]). *Let $\beta \in \Delta_{\text{im}}$ be non-isotropic. Let $\alpha \in \bar{\Delta}$ be such that $\alpha + \mathbb{N}\beta \subset \bar{\Delta}$. Then $R_\alpha(\beta)$ has exponential growth in β.*

When β is isotropic, [Mar21, Corollary C] and [Kac90, Corollary 9.12] no longer apply. However, we observe that when $(\alpha, \beta) = 0$, it is essentially a case of affine Kac–Moody algebras.

Theorem 2.3 ([CCM+24]). *Let $\beta \in \Delta_{\text{im}}$ be isotropic, and $\alpha \in \bar{\Delta}$ be such that $R_\alpha(\beta)$ is non-trivial. Suppose $(\alpha, \beta) = 0$. Then $R_\alpha(\beta)$ is always bi-infinite, and*

(1) *If α is real, $R_\alpha(\beta)$ consists only of real roots which have multiplicity 1.*
(2) *If α is imaginary, $R_\alpha(\beta)$ consists of imaginary roots only, and the multiplicities take on at most 3 values, at most 2 of which occur periodically.*

In particular, part (2) is derived from a case by case study of untwisted and twisted affine Kac–Moody algebras.

If $(\alpha, \beta) \neq 0$, then the trick of reducing to the affine case no longer applies. Yet we can construct a subalgebra in $\bigoplus_{k\in\mathbb{N}} \mathfrak{g}_{k\beta}$ that is isomorphic to the Heisenberg algebra. We denote it as $H(\beta)$. Then by applying the representation theory of $H(\beta)$, we show that there exists a subspace in $\bigoplus_{k\in\mathbb{N}} \mathfrak{g}_{\alpha+k\beta}$ which is isomorphic to an irreducible induced module of $H(\beta)$.

Theorem 2.4 ([CCM+24])**.** *Let $\beta \in \Delta_{\mathrm{im}}$ be isotropic, and $\alpha \in \bar{\Delta}$ be such that $R_\alpha(\beta)$ is non-trivial. Suppose $(\alpha, \beta) \neq 0$. Then $R_\alpha(\beta)$ is semi-infinite, and has subexponential growth in β.*

We also provide the following "local" property regarding the multiplicities.

Theorem 2.5. *If α and β are distinct roots of $\mathfrak{g}$ such that $(\alpha, \beta) < 0$, then*

$$\dim \mathfrak{g}_{\alpha+\beta} \geq \dim \mathfrak{g}_\alpha + \dim \mathfrak{g}_\beta - 1.$$

2.3 Future Directions We hope to answer the following questions:

(1) Are the lower bounds provided in the above table strict?
(2) For non-isotropic β, what is the structure of $\mathfrak{S}_\alpha(\beta)$ as an $\mathfrak{sl}(2)$-module? Here this $\mathfrak{sl}(2)$ is constructed using root vectors in $\mathfrak{g}_{\pm\beta}$.
(3) What can we say about the structure of $\mathfrak{S}_\alpha(\beta)$ when β is isotropic?

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